11th Practice sheet for the lecture: Combinatorics (DS I)

Felsner/ Schröder

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Due dates: 11th of July

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

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(1) A design $(\mathcal{P}, \mathcal{B})$ is *resolvable*, if there is a partition of $\mathcal{B}$ into partitions of $\mathcal{P}$.
Let $(\mathcal{P}, \mathcal{B}) = S(2, n+1, n^2+n+1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that

$$\Big(\mathcal{P} \setminus B, \{C \setminus B \mid C \in \mathcal{B} \setminus \{B\}\}\Big)$$

is a resolvable $S(2, n, n^2)$ design.

(2) Consider red-blue-face-colorings of the platonic solids with rotational symmetries.

(a) Determine the number of differently colored dodecahedra.

(b) Determine the number of differently colored icosahedra.

(c) Determine the number of differently colored soccer balls. Note that a soccer ball is a truncated icosahedron.

(3) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a relabeling of the vertices of G yields the graph H, i.e., if there exists a bijection $\Phi : V \to V'$ such that the mapping $\Psi(\{v, w\}) := \{\Phi(v), \Phi(w)\}$ is a bijection from E to E'.
Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges. Let $S_n^{[2]}$ be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi^{[2]}(\{i, j\}) := \{\pi(i), \pi(j)\}$, $\pi \in S_n$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_{S_n^{[2]}}\Big(1 + x, 1 + x^2, \ldots 1 + x^{\binom{n}{2}}\Big).$$

How can you derive the type of a permutation $\pi^{[2]}$ based on the type of π?

(4) Let $(\mathcal{P}, \mathcal{B})$ be a $S_\lambda(t, k, v)$. Let $I, J \subset \mathcal{P}$ with $|I| = i, |J| = j, I \cap J = \emptyset$. Let the number of blocks in $\mathcal{B}$ that contain I and are disjoint with J be $\lambda_{I,J}$. Prove that

$$\lambda_{i,j} := \lambda_{I,J} = \lambda \frac{\binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}$$

(5) Let there be 27 cubes of volume 1. Count the number of ways to paint the faces of the cubes with 3 colors, such that for any color c, you can stick the cubes together to form a cube of volume 27 whose faces are of color c, up to rotational symmetries of the 27 cubes it consists of and their order.