10th Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 28th of June 2023
Due dates: 3rd/4th of July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

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(1) Euler Phi-function

(a) Prove that for m, n with $\gcd(m, n) = 1$ it holds that $\phi(m \cdot n) = \phi(m) \cdot \phi(n)$.

(b) For a prime p and an integer k, determine $\phi(p^k)$.

(2) Consider necklaces with 12 beads of at most three different colors

(a) How many different necklaces exist?

(b) How many different necklaces with 3 red, 4 green, and 5 blue beads exist?

(3) Popular matchings
Let G be a bipartite graph and each vertex v has a strict preference list L_v on its neighbors. We compare the two matchings by comparing votes of the vertices. Let M and M' be two matchings of G. Each vertex v gives a vote for (M, M')

$$vote_v(M, M') = \begin{cases} -1, & v \text{ prefers } M'(v) \text{ over } M(v) \\ 0, & v \text{ has no prefered matching, e.g.,} M(v) = M'(v) \\ 1, & v \text{ prefers } M(v) \text{ over } M'(v) \end{cases}$$

We say M is *more popular* than M' if $\sum_v vote_v(M, M') > 0$ and write $M' \prec M$. We say M is *popular* if no matching M' exists that is more popular than M.

(a) Show that there are graphs with preference lists without a popular matching in general graphs/ in the one-sided model, where only one side of the bipartition has a preference list.

(b) Show that every stable matching is popular.

(c) Show that popular matchings may be strictly larger than stable matchings.

(*) Finding max-size popular matchings can be done by a modified Gale-Shapely. Inform yourself about the algorithm. (A rough idea of the proof is fine.)

(*) Consider an $n \times n$ chess–board for even $n \in \mathbb{N}$. How many configurations (up to the symmetries of D_4) of n rooks (Türme) on the board are there, such that no rook can capture another one? (Hint: Use the CFB Lemma) How many distinct configurations exist, if you are only considering the symmetries of D_4, which map black squares to black squares?