5. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 23. May 2023

Due dates: 29.-30. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

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(1) We say a vertex in a binary tree is an *inner* vertex if it has at least one child. Let t_n denote the number of binary trees with n inner vertices. Use the symbolic method to determine the generating function $T(z)$.

(2) A set of chords of a convex $2n$-gon is a *quadrangulation* if no two chords intersect and all faces are quadrangles. Let a_n denote the number of quadrangulations of a convex $2n$-gon. Use the symbolic method to find the generating function $A(x) = \sum_{n \geq 0} a_n x^n$. [Hint: Find a connection to ternary trees.]

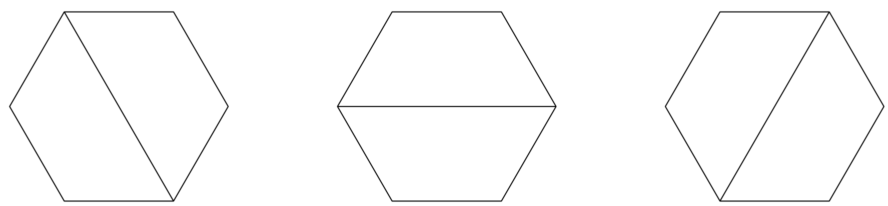

Figure 1: The 3 quadrangulations of a 6-gon

(3) On a table there are 100 coins. A and B are going to remove coins from the table by turns. In each turn they can remove 2, 5 or 6 coins. The first one that cannot make a move loses. Determine who has a winning strategy if A plays first.

(4) Prove the following facts about the Fibonacci numbers F_n.

(a) Every $n \in \mathbb{N}$ can be partitioned uniquely into different Fibonacci numbers $F_k, k > 1$, such that no two consecutive Fibonacci numbers are used.

(b) $F_{2n} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} F_k$ [Hint: There is a nice bijection using monominoes.]

(5) A permutation $\pi \in S_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \ldots$ holds. Let $\mathrm{Alt}_n \subseteq S_n$ be the set of alternating permutations. A permutation σ is *reverse alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$ holds. Let $\mathrm{RAlt}_n \subseteq S_n$ be the set of reverse alternating permutations.

(a) Prove $|\mathrm{Alt}_n| = |\mathrm{RAlt}_n|$.

(b) Let $E_n := |\mathrm{Alt}_n|$ and prove $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for all $n \geq 1$.
[Hint: Apply (a)]

(c) Let $E_n(q) := \sum_{\pi \in \mathrm{RAlt}_n} q^{\mathrm{inv}(\pi)}$ and $E_n^{\star}(q) := \sum_{\pi \in \mathrm{Alt}_n} q^{\mathrm{inv}(\pi)}$. Prove

$$E_n^{\star}(q) = q^{\binom{n}{2}} E_n\left(\frac{1}{q}\right).$$