3. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 4. Mai 2023
Due dates: 15. /16. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

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(1)
  (a) How many subsets of the set $[n]$ contain at least one odd integer?
  (b) How many multi-subsets of the set $[n]$ have size k? (Multi(-sub-)sets may contain the same element more than once.)
  (c) Let $\subseteq$ denote the (non-strict) subset relation. For a given $k \in [n]$, how many sequences $(T_1, T_2, \ldots, T_k)$ are there with

$$\emptyset \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_k \subseteq [n] \; ?$$

(2) Show that the number of partitions of n where no part is divisible by $d \in \mathbb{N}\backslash\{0,1\}$ equals the number of partitions of n where no d parts have the same size.

(3) Let there be n points in the plane with integer coordinates. Among them, we want to find three points p_1, p_2, p_3, such that their barycenter $b = \frac{1}{3}(p_1 + p_2 + p_3)$ has integer coordinates.
  (a) Prove that if $n \geq 13$, we can always find such 3 points.
  (b) Prove that if $n = 9$, we can always find such 3 points.
  (c) Find a set of $n = 8$ points, where it is impossible to choose 3 such points.

(4) $c(n,k)$ denotes the number of permutations of $[n]$ with k cycles and $S(n,m)$ the number of partitions of $[n]$ into m parts. Show the following identities:

$$c(n+1, m+1) = \sum_{k=m}^{n} \binom{k}{m} c(n,k)$$

$$S(n+1, k+1) = \sum_{i} \binom{n}{i} S(i,k)$$

$$\sum_{k=0}^{n} c(n, n-k) x^k = \prod_{k=0}^{n-1} (1 + kx)$$

$$\sum_{k=0}^{\infty} S(n+k, n) x^k = \prod_{k=1}^{n} \frac{1}{1-kx}$$

(5) In the parliament of some country there are $2n+1$ seats filled by 3 parties. How many possible distributions (i,j,k), (i.e. party 1 has i, party 2 has j, and party 3 has k seats) are there, such that no party has an absolute majority? Give a nice combinatorial proof. (Hint: Look at small numbers and make a good guess.)