2. Practice sheet for the lecture: **Felsner/ Schröder**

Combinatorics (DS I) 2. Mai 2023

Due dates: 8.-9. Mai

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

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(1) Let $x^{\underline{n}} := x \cdot (x+1) \cdots (x+n-1)$ denote the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}} \quad \left[\text{Hint: } \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}\right]$$

(2) Find a closed form expression for A_n, $n \in \mathbb{N}$:

$$A_n = \sum_{k=0}^{n} \binom{n-k}{k} (-1)^k.$$

(3) A permutation $\pi \in S_n$ is a transposition if the position of exactly two elements is switched, i.e. in cycle notation there are $n-2$ fixpoints and a cycle of length 2.

a) Show that each permutation $\pi \in S_n$ is a composition of transpositions. We denote the minimum number of transpositions needed to express π by $t(\pi)$.

b) Let $\pi \in S_n$ be a permutation with exactly one cycle of length k and $n-k$ fixed points. Show that

$$t(\pi) = k - 1$$

(c) What is the expected number of left-to-right-maxima in a random permutation? What is the expectation of $t(\pi)$?

(4) There are nine caves in a forest, each being the home of one animal. Between any to caves there is a path, but they do not intersect. Since it is election time, the candidating animals travel through the forest to advertise themselves. Each candidate visits every cave, starting from their one cave and arriving there at the end of the tour again. Also each path is only used once in total, since nobody wants to look at the embarrassing posters (not even their own), put there on the first traverse. Under these assumptions, what is the maximum possible number of candidates?

(5) Mutually orthogonal latin squares (MOLS)

(a) Prove one part of the unproved theorem from the lecture: Construct $(n-1)$ MOLS of order n from a projective plane of order n.

(b) Let $\mathbb{F}$ be a field of n elements. For all $q \in \mathbb{F}\backslash\{0\}$, define $n \times n$ tables Q_q by $Q_q(x, y) = qx + y$. Show that the tables of this family are MOLS.