1. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Schröder**, 22. April 2023

Due dates: 2. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

WARMUP: Find a recursive formula for the series $\sum_k \binom{n-k}{k}$.

(1) A spider has a particular sock and a particular shoe for each of his eight feet. In how many different orders can it put on his shoes and socks, assuming that on each foot it has to put on the sock first?

(2) The squares of a 4×7 chessboard are coloured arbitrarily black and white (i.e. there are $2^{4 \cdot 7}$ colourings). Show that there is an $i \times j$-rectangle with $i \geq 2$ and $j \geq 2$, such that all four of its corners have the same colour. Is the same true for a 4×6 chessboard?

(3) Consider a chess tournament of n players, each playing against every other participant. Show that at each point of time during the tournament there exist at least two players having finished the same number of games.

(4) A sequence of numbers $a_1, a_2, \ldots, a_n$ is called *unimodal,* if there exists an $m \in [n]$, such that $a_i \leq a_{i+1}$ for all $i < m$ and $a_i \geq a_{i+1}$ for all $i \geq m$. Give three different proofs of the unimodality of the sequence $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ for all $n \in \mathbb{N}$, based on the three given hints:

   (a) Use the definition
   $$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!}.$$

   (b) Consider the recursive definition
   $$\binom{n}{k} := \binom{n-1}{k-1} + \binom{n-1}{k} \text{ and } \binom{n}{0} = \binom{n}{n} = 1,$$
   based on Pascal's triangle.

   (c) Use the bijection between $\binom{n}{k}$ and the number of subsets of $[n]$, having k elements.

(5) Permutations

   (a) Give an alternative proof for the generating function of derangements (6)
   $$D(z) := \sum_{n=1}^{\infty} \frac{d(n)}{n!} z^n = \frac{e^{-z}}{1-z}$$
   by manipulating $D(z)e^z$ and using an appropriate identity from the lecture.

   (b) Prove the following identity:
   $$d(n) = n \cdot d(n-1) + (-1)^n$$

   (c) How many permutations in S_n are derangements and involutions?
   (Let X be a set. A function $f : X \to X$ is an *involution* if $f(f(x)) = x \; \forall x \in X$.)

   (d) What is the probability that a random involution is a derangement as n approaches infinity? In particular, is this probability well-defined?