0. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 12. April 2023

Due dates: 24./25. April (optional)
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI23.html`

---

(1) (OIM shortlist 2009) On an 8×8 board there is a lamp in every square. Initially every lamp is turned off. In a move we choose a lamp and a direction (it can be the vertical direction or the horizontal one) and change the state of that lamp and all its immediate neighbors in that direction (in one move, the state of at most 3 lamps is changed). After a certain number of moves, there is exactly one lamp turned on. Find all the possible positions of that lamp.

(2) (IMO 2000) A magician has cards numbered from 1 to 100, distributed in 3 boxes of different colors so that no box is empty. His trick consists in letting one person of the crowd choose two cards from different boxes without the magician watching. Then the person tells the magician the sum of the numbers on the two cards and he has to guess from which box no card was taken.

a) Find a distribution of the cards to the boxes so that the trick of the magician always works.

b) In how many ways can the magician distribute the cards so that his trick always works?

(3) Show that if n is a non-negative integer, then

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

(4) (Estonia 2007) An exam with k question is presented to n students. A student fails the exam if he gets less than half the answers right. We say that a question is easy if more than half of the students get it right. Decide if it is possible that

a) All students fail even though all the questions were easy.

b) No student fails even though no question was easy.

(5) (Japan 2009) Let N be a positive integer. Suppose that a collection of integers was written in a blackboard so that the following properties hold:

- each written number k satisfies $1 \leq k \leq N$;
- each k with $1 \leq k \leq N$ was written at least once;
- the sum of all written numbers is even.

Show that it is possible to label some numbers with $\circ$ and the rest with $\times$ so that the sum of all numbers with $\circ$ is the same as the sum of all numbers with $\times$.

(6) (Great Britain 2007) In Hexagonia, there are six cities connected by railways in such a way that between every two cities there is a direct railway. On Sundays, some railways close for repairs. The train company guarantees that passengers will be able to get from any city to any other city (it may not be in a direct way) at any time. In how many ways can the company close railways so that this promise is kept?