13th Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 6th of July 2021
Due dates: 13th/15th of July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1) How many permutations of the 26 letters of the English alphabet do not contain any of the strings `white`, `black` or `pink`?

(2) In the lecture, we deduced the formula $N(\emptyset) = \sum_B (-1)^{|B|} N_{\geq}(B)$. Show that it implies the following version of the *inclusion-exclusion formula*. For $A_1, \ldots, A_r \subseteq X$, it holds:

$$\left| X - \bigcup_{i=1}^{r} A_i \right| = \sum_{I \subseteq [r]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

(3) (This exercise gives 2 points.) For a prime power q, consider the poset P of all subspaces of the n-dimensional vector space $V_n(q)$ over $\mathbb{F}_q$ with the subspace relation.

(a) Show the following identity using one of the q-binomial theorems:

$$\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} (-1)^i q^{\binom{i}{2}} = 0$$

(b) Compute the Möbius function of P.

(c) Count the number of linear functions from $V_n(q)$ *onto* $V_k(q)$, that is, the number of surjective linear functions $f : V_n(q) \to V_k(q)$.

(4) Use Möbius inversion to show that for every positive integer n, it holds

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$$

[Hint: Recall that $\sum_{d|n} \phi(d) = n$.]

(*) Let $D = (\{v_1, \ldots, v_n, s, t\}, \vec{E})$ be a directed graph. We are interested in the number of *Hamilton paths* from s to t, that is, the number of paths from s to t that contain all the $v_1, ..., v_n$ as inner vertices exactly once. Develop a formula which can be evaluated in $o(n!)$ time. [Hint: You can use Möbius inversion.]