12th Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 30th of June 2019
Due dates: 6th/8th of July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1) (This exercise gives 2 points.) For which of the parameter sets does a design exist? Either show that there is none or present one. (if λ is omitted, it is 1.)

(a)	$S(2, 3, 127)$	(d)	$S(2, 7, 36)$
(b)	$S_2(4, 7, 13)$	(e)	$S_{12}(2, 16, 21)$
(c)	$S_3(2, 10, 25)$	(f)	$S_3(3, 5, 21)$

[Hint to (f): In the edge set of K_7, use the cycle C_5, the star and more as blocks.]

(2) Let $(\mathcal{P}, \mathcal{B})$ be a $S_\lambda(t, k, v)$ design.

(a) Let $p \in \mathcal{P}$ and $\mathcal{B}^p := \{B : p \notin B \in \mathcal{B}\}$ be the set of blocks, which do not contain p. Show that $(\mathcal{P} \setminus \{p\}, \mathcal{B}^p)$ is a design. What are its parameters?

(b) Consider the *complement* of a $S_\lambda(t, k, v)$, i.e., replace each block by its complement. Prove that the complement of a $S_\lambda(t, k, v)$ is a design for the same parameter t. Determine its other parameters.

(3) A design $(\mathcal{P}, \mathcal{B})$ is *resolvable*, if there is a partition of $\mathcal{B}$ into partitions of $\mathcal{P}$.
Let $(\mathcal{P}, \mathcal{B}) = S(2, n + 1, n^2 + n + 1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that

$$\Big(\mathcal{P} \setminus B, \big\{C \setminus B \mid C \in \mathcal{B} \setminus \{B\}\big\}\Big)$$

is a resolvable $S(2, n, n^2)$ design.

(*) We will call a design a *circle-design*, if $\mathcal{P}$ can be represented as a point set in the plane and $\mathcal{B}$ can be represented as a set of circles in the plane, such that the incidence structure is given by a point lying on a circle. For which values of the parameter t do there exist non-trivial circle designs?

(5) A *perfect matching* of K_{2n} is a set of n disjoint edges. A *resolution* of K_{2n} is a partition of its edge set into perfect matchings.

(a) Show that two disjoint perfect matchings of K_6 determine a unique resolution. [Hint: Note that the union of two perfect matchings of K_6 is a cycle.]

(b) Consider $K_6 = (V, E)$. Let $\mathcal{P}$ denote the set of its perfect matchings and $\mathcal{R}$ the set of its resolutions. For $e \in E$ we define $B_e := \{v \in V | v \in e\} \cup \{P \in \mathcal{P} | e \in P\}$ and for each $R \in \mathcal{R}$, we define $B_R := \{P \in \mathcal{P} | P \in R\}$. Conclude that the following is a $S(2, 5, 21)$:

$$(\mathcal{P} \cup V, \{B_e | e \in E\} \cup \{B_R | R \in \mathcal{R}\})$$

(*) Show that the design from (b) is a projective plane.