**11th Practice sheet for the lecture:** **Felsner/ Schröder**
**Combinatorics (DS I)** 22nd of June 2021
Due dates: 29th of June/ 1st of July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1) Euler Phi-function

  (a) Prove that for $m, n$ with $\gcd(m, n) = 1$ it holds that $\phi(m \cdot n) = \phi(m) \cdot \phi(n)$.

  (b) For a prime $p$ and an integer $k$, determine $\phi(p^k)$.

(2) Consider necklaces with 12 beads of at most three different colors

  (a) How many different necklaces exist up to equivalence by the dihedral group?

  (b) How many different necklaces with 3 red, 4 green, and 5 blue beads exist?

(3) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a relabeling of the vertices of $G$ yields the graph $H$, i.e., if there exists a bijection $\Phi : V \to V'$ such that the mapping $\Psi(\{v, w\}) := \{\Phi(v), \Phi(w)\}$ is a bijection from $E$ to $E'$.
Let $g_{n,k}$ be the number of non-isomorphic graphs on $n$ vertices with $k$ edges. Let $S_n$ be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i, j\}) := \{\pi(i), \pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_{S_n}\Big(1 + x, 1 + x^2, \ldots 1 + x^{\binom{n}{2}}\Big).$$

(*) Consider an $n \times n$ chess–board for even $n \in \mathbb{N}$. How many configurations (up to the symmetries of $D_4$) of $n$ rooks (Türme) on the board are there, such that no rook can capture another one? (Hint: Use the CFB Lemma) How many distinct configurations exist, if you are only considering the symmetries of $D_4$, which map black squares to black squares?

(5) Let there be 27 cubes of volume 1. Count the number of ways to paint the faces of the cubes with 3 colors, such that for any color $c$, you can stick the cubes together to form a cube of volume 27 whose faces are of color $c$, up to rotational symmetries of the 27 cubes it consists of and their order.