10th Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 17th of June 2021

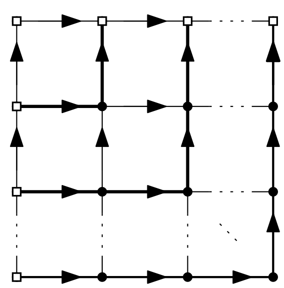
Due dates: 22nd/24th of June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

(1) Count the number of triples (p_0, p_1, p_2), where p_i are vertex-disjoint lattice paths of length 6 from $A_i = (i, 2-i)$ to $B_i = (3+i, 5-i)$ (for $i \in \{0,1,2\}$).

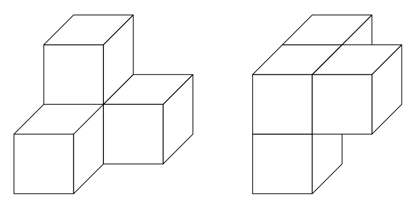
(2) (Weakness of homotopy)
Show that T-tetrominoes cannot tile the 2×4 rectangle, but that the boundary word of this rectangle is in the group generated by the boundary words of T-tetrominoes. [Hint: Combine the words of two T-tetrominoes to show that $xy\bar{x}\bar{y} = \bar{x}\bar{y}xy$. Use this fact to move a misplaced square to another place.]

(3) Apply the Lemma of Lindström-Gessel-Viennot in order to prove the formula for the Vandermonde determinant:

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j)$$



(a) [Hint: Find weights for the horizontal edges of this 'lattice'-graph and its unique vertex disjoint path system.]



(b) Two rotations of the 2-tetrahedron

Figure 1: Illustration of exercises 3 and 4

(4) Let a *n-tetrahedron* be the union of all unit cubes with integer coordinates, such that the sum of all coordinates of the least corner of the cube is at most $n-1$. Figure 1b shows an example. For which values of $n \le 14$ can n-tetrahedra be tiled by 2-tetrahedra? Give a lower bound for the number of such tilings which is at least exponential in n.

(5) (*) Prove the following identity:

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n$$

[Hint: Consider grid paths. Let the weight of a grid path be the number of times it touches the diagonal (k,k).]