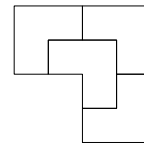(1)   Theorem of Hall

    (a)   Find an infinite counterexample to the Theorem of Hall, i.e., find a bipartite graph $G = (X \cup Y; E)$ with the property that $|N(S)| \geq |S|$ for all $S \subset X$ and there is no matching containing all vertices of X.

    (b)   The analogue of the Hall condition for general graphs is the *Tutte condition*. Inform yourself about this condition and show at least one implication of Tutte's theorem.

(2)   A *T-tetromino* is a tile consisting of exactly 4 squares, one of them adjacent to the others, see Figure 1a.

    (*)   Which $n \times n$ squares can be tiled by T-tetrominoes?

    (b)   What is the homology group of the T-tetromino?
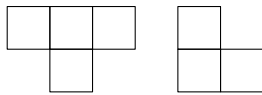
(a) T-Tetromino and L-Triomino   (b) A scaled L-triomino tiled by L-triominoes

Figure 1: Illustration of exercises (2) and (3)

(3)   A *L-triomino* is a consisting of exactly 3 squares, one of them adjacent to the others, see Figure 1a.

    (a)   Which $n \times n$ squares can be tiled by L-triominoes?

    (b)   Which squares with a removed corner square can be tiled by L-triominoes?

    (c)   If we scale an L-triomino by a positive integer, it is a *scaled L-triomino*. Which scaled L-triominoes can be tiled by L-triominoes?

(4)   Let $P = (X, \leq)$ be a poset. We call a chain decomposition $\{C_i\}_i$ of P *greedy chain decomposition (GCD)* if it has the following property: C_1 is a maximum chain in P, and for $i > 1$, C_i is a maximum chain in P_i where P_i is the subposet of P induced by $X - \bigcup_{j<i} C_j$.
Prove or disprove: $\exists c \in \mathbb{R}$ such that any GCD of any finite poset P has size at most $c \cdot w$, where w is the width of P.

(5)   Consider two magicians M_1, M_2 in well separated rooms. A volunteer picks five cards from a standard deck (52 cards) and hands them to M_1. M_1 keeps one of the five cards and puts the other four (in specific order) in an envelope. The envelope is brought to M_2 who opens it, has a look at the cards and announces the fifth card.

    (a)   Explain the existence of a strategy for this trick with the aid of Hall's Theorem.

    (*)   Find a playable strategy.