
**8. Practice sheet for the lecture:
Combinatorics (DS I)**

Felsner/ Schröder
01. June 2021

Due dates: 08./10. June

<http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html>

- (1) Symmetric chain decompositions of \mathcal{B}_n
 - (a) Given a symmetric chain C in \mathcal{B}_n , is there always a symmetric chain decomposition containing C ?
 - (b) Show that the number of chains of length $n + 1 - 2k$ in a symmetric chain decomposition of \mathcal{B}_n is $\binom{n}{k} - \binom{n}{k-1}$.
 - (c) Use (1b) to derive the known explicit formula for the Catalan numbers, i.e. $C_n = \frac{1}{n+1} \binom{2n}{n}$.
(Hint: Valid bracket expressions are counted by Catalan numbers.)
- (2) Let \mathcal{B}_n^\vee be the truncation of the Boolean lattice where the maximal and minimal element is deleted. Let \mathcal{C} be a symmetric chain decomposition which is canonical (originating from the bracketing process). Let $\bar{\mathcal{C}}$ be its complement, i.e. for a chain $C \in \mathcal{C}$ the set \bar{C} of complements of sets in C is a set in $\bar{\mathcal{C}}$.
 - (a) Show that $\bar{\mathcal{C}}$ is a symmetric chain decomposition.
 - (b) Show that \mathcal{C} and $\bar{\mathcal{C}}$ are *orthogonal*, i.e. $|C \cap D| \leq 1$ for all $C \in \mathcal{C}$ and $D \in \bar{\mathcal{C}}$.
- (3) Let \mathcal{A} be a family of k -subsets of $[n]$ and \mathcal{B} be a family of l -subsets of $[n]$ such that $l + k \leq n$ and $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
 - (a) Show that $|\mathcal{A}| < \binom{n-1}{k-1}$ or $|\mathcal{B}| \leq \binom{n-1}{l-1}$.
Hint: Use shadows as in the second proof of the Erdős-Ko-Rado theorem.
 - (b) Deduce the Erdős-Ko-Rado theorem from (a).
- (4) Prove or disprove: Let $P = (X, \leq)$ be a poset and C a maximum chain. Then the width of $P' := (X \setminus C, \leq)$ is smaller than the width of P . What if C is a maximal chain? Can this be used to prove Dilworth's theorem?
- (5) Counting chains and antichains in \mathcal{B}_n
 - (a) What is the number $C_n^{(k)}$ of chains of size k in the boolean lattice?
 - (b) Prove by bijection that the number of antichains of size 2 is $\frac{1}{2}C_n^{(3)}$.
 - (c) What is the number of antichains of size 3 in the boolean lattice?