
**7. Practice sheet for the lecture:
Combinatorics (DS I)**

Felsner/ Schröder
27. May 2021

Due dates: 1.-3. June

<http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html>

- (1) Prove the following identity:

$$\sum_{k=0}^n \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n$$

[Hint: Consider grid paths. Let the weight of a grid path be the number of times it touches the diagonal (k, k) .]

- (2) Let $P := (M, \leq)$ be a finite poset with n elements, i.e. $|M| = n$. For a given ordering of the elements (m_1, \dots, m_n) , we can associate with P a relation matrix $(a_{ij}) = A \in \{0, 1\}^{n \times n}$, such that $a_{ij} = 1 \Leftrightarrow m_i \leq m_j$.

(a) Show that it is possible to find an ordering on (m_1, \dots, m_n) such that A is an upper triangle matrix.

(b) Show that $\#\{(i, j) \mid a_{ij} = 1\} = \binom{n+1}{2} \Leftrightarrow P$ is a total order.

(c) Find matrix properties for a $\{0, 1\}$ -matrix B , which imply that B represents a partial order relation.

- (3) A poset $P = (X, \leq)$ is *ranked* if there exist a function $r : X \rightarrow \mathbb{N}$ such that for all $x \in X$, the rank $r(x)$ is the length of *all* maximal chains ending in x .

For $n \in \mathbb{N}$, the *divisor-poset* P_n is the set of all divisors of n ordered by divisibility:

$$P_n := \{x \in \mathbb{N} : x \mid n\}, \{(x, y) \in \mathbb{N}^2 : x \mid y \text{ and } y \mid n\}.$$

Prove that P_n is ranked. Sketch the Hasse-Diagramm of P_{60} and compute its ranks. Is P_n a lattice?

- (4) q -analogue

(a) How many different maximal chains of nested subspaces of $V_n(q)$ exist?

(b) Define the q -analogue of the boolean lattice. Consider an antichain \mathcal{A} and show that the q -analogue of the LYM inequality holds:

$$\sum_{k=0}^n \frac{p_k(\mathcal{A})}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1$$

- (5)

(a) Show that every finite poset is isomorphic to the containment order on some family of sets.

(b) Show that every finite poset can be obtained by taking a subset of the integers $I \subseteq \mathbb{N}$ together with the divisor relation \mid .

(c) Find a poset with an element that is greater than every other element and another element, which is smaller than every other element, but which is not a lattice.