6. Practice sheet for the lecture: **Felsner/ Schröder**

Combinatorics (DS I) 19. May 2021

Due dates: 25.-27. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1) The q–binomials fulfill the equations

(a)

$$\sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{i-k}$$

(b)

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via two different models of q-binomials.

(2)

(a) Show this q-binomial theorem for $x, y \in \mathbb{C}, n \in \mathbb{N}$:

$$\prod_{k=0}^{n} (x + q^k y) = \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix} y^k x^{n-k}$$

(b) Show that the q–binomials fulfill the equation

$$\sum_{k \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} z^k = \prod_{i=0}^{n} \frac{1}{1 - q^i z}$$

(3) Search in the literature for at least three different combinatorial structures that are also counted by Catalan numbers, one of which (at least) is not from Wikipedia, and prove that they are. (You are allowed to present proofs from the literature you read.)

(4) Let $\mathcal{B}$ be a set with a certain weight function $|\cdot| : \mathcal{B} \to \mathbb{N}$ and let $\mathcal{A}$ be the set of all finite subsets of $\mathcal{B}$. Show that the generating function A of $\mathcal{A}$ can be expressed as

$$A(z) = \prod_{n \geq 0} (1 + z^n)^{b_n}$$

Recall that $A(z) = \sum_{a \in \mathcal{A}} x^{|a|} = \sum_{n \geq 0} a_n x^n$ and, likewise, $B(z) = \sum_{b \in \mathcal{B}} x^{|b|} = \sum_{n \geq 0} b_n x^n$. How is the weight of $a \in \mathcal{A}$ related to the weight defined on $\mathcal{B}$?

(5) (Colombia 2011) Ivan and Alexander write lists of integers. Ivan writes all the lists of length n with elements $a_1, \ldots, a_n$ such that $|a_1| + \cdots + |a_n| \leq k$. Alexander writes all the lists with length k with elements $b_1, \ldots, b_k$ such that $|b_1| + \cdots + |b_k| \leq n$. Prove that Alexander and Ivan wrote the same number of lists.