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**5. Practice sheet for the lecture:  
Combinatorics (DS I)**

**Felsner/ Schröder**  
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Due dates: 18.-20. May

<http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html>

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- (1) Let  $t_n$  denote the number of binary trees with  $n$  inner vertices. We say a vertex in a binary tree is an *inner* vertex if it has children. Use the symbolic method to determine the generating function.
- (2) A set of chords of a convex  $2n$ -gon is a *quadrangulation* if no two chords intersect and all faces are quadrangles. Let  $a_n$  denote the number of quadrangulations of a convex  $2n$ -gon. Use the symbolic method to find the generating function  $A(x) = \sum_{n \geq 0} a_n x^n$ . [Hint: Find a connection to ternary trees.]

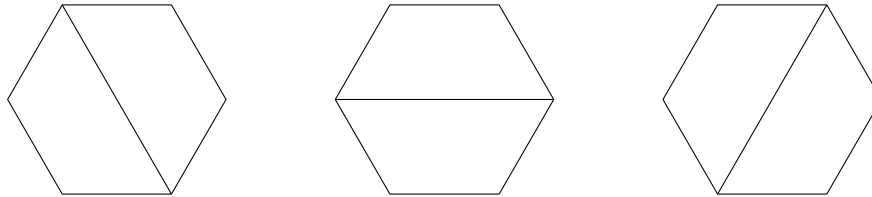


Figure 1: The 3 quadrangulations of a 6-gon

- (3) On a table there are 100 coins. A and B are going to remove coins from the table by turns. In each turn they can remove 2, 5 or 6 coins. The first one that cannot make a move loses. Determine who has a winning strategy if A plays first.
- (4) Prove the following facts about the Fibonacci numbers  $F_n$ .
  - (a) Every  $n \in \mathbb{N}$  can be partitioned uniquely into different Fibonacci numbers  $F_k, k > 1$ , such that no two consecutive Fibonacci numbers are used.
  - (b)  $F_{2n} = \sum_{k=0}^n \binom{n}{k} F_k$  [Hint: There is a nice bijection using monominoes.]
- (5) A permutation  $\pi \in S_n$  is *alternating* if  $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \dots$  holds. Let  $\text{Alt}_n \subseteq S_n$  be the set of alternating permutations. A permutation  $\sigma$  is *reverse alternating* if  $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \dots$  holds. Let  $\text{RAlt}_n \subseteq S_n$  be the set of reverse alternating permutations.
  - (a) Prove  $|\text{Alt}_n| = |\text{RAlt}_n|$ .
  - (b) Let  $E_n := |\text{Alt}_n|$  and prove  $2E_{n+1} = \sum_{k=0}^n \binom{n}{k} E_k E_{n-k}$  for all  $n \geq 1$ . [Hint: Apply (a)]
  - (c) Let  $E_n(q) := \sum_{\pi \in \text{RAlt}_n} q^{\text{inv}(\pi)}$  and  $E_n^*(q) := \sum_{\pi \in \text{Alt}_n} q^{\text{inv}(\pi)}$ . Prove

$$E_n^*(q) = q^{\binom{n}{2}} E_n\left(\frac{1}{q}\right).$$