**4. Practice sheet for the lecture:** **Felsner/ Schröder**
**Combinatorics (DS I)** 5. Mai 2021

Due dates: 11.-13. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1) Let $a_k$ denote the number of words of length $k$ over the alphabet $\{u, l, r\}$ with no $l$ and $r$ consecutive, i.e. $lr$ and $rl$ do not appear. These words can be interpreted as grid paths of length $k$ which go up, left and right, and do not intersect themselves.

(a) Find a linear recursion for $a_k$.

(b) Express the generating function as a rational function.

(c) Find a closed form for $a_k$.

(d) Give a combinatorial proof for the equation:

$$A(z) = \left(1 + 2\sum_{k=1}^{\infty} z^k\right) \cdot (z \cdot A(z) + 1)$$

(2) In how many ways can you pay $n$ Dollar with 1\$, 5\$ and 10\$ notes? Find a generating function and compute the number of ways to pay 50 Dollar.
[Hint: find three distinct generating functions, each for one type of notes only, and put them together in the right way]

(3) Let $F(z) := \sum \frac{B(n)}{n!} z^n$ be the exponential generating function of the Bell numbers $B(n)$. Show that there exist a function $f(x)$ such that

$$F'(x) = f(x)F(x).$$

Solve the differential equation and deduce a closed formula for $F(x)$. Use the recursion for the Bell numbers $B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(n-k)$

(4) Consider a tower of size $2 \times 2 \times n$ and bricks of size $2 \times 1 \times 1$. How many different tilings of the tower with bricks (and rotated copies) exist?
[Hint: Consider both, the number of tilings $a_k$ of a tower of height $k$ and the number of unfinished tilings $b_k$, where in the top level only 2 cubes (of the 4) are covered by standing bricks.]

(5) (Germany 2004) Given a graph with black and white vertices, a movement consists in changing the color of a vertex and every other vertex adjacent to it. If we start with any graph G with only white vertices, is it always possible to turn them all black with such movements?