3. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 28. April 2021

Due dates: 4. /6. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1)
  (a) How many subsets of the set $[n]$ contain at least one odd integer?

  (b) How many multi-subsets of the set $[n]$ have size k? (Multi(-sub-)sets may contain the same element more than once.)

  (c) Let $\subseteq$ denote the (non-strict) subset relation. For a given $k \in [n]$, how many sequences $(T_1, T_2, \ldots, T_k)$ are there with

$$\emptyset \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_k \subseteq [n] \; ?$$

(2) Show that the number of partitions of n where no part is divisible by $d \in \mathbb{N} \setminus \{0, 1\}$ equals the number of partitions of n where no d parts have the same size.

(3) Let there be n points in the plane with integer coordinates. Among them, we want to find three points p_1, p_2, p_3, such that their barycenter $b = \frac{1}{3}(p_1 + p_2 + p_3)$ has integer coordinates.

  (a) Prove that if $n \geq 13$, we can always find such 3 points.

  (b) Prove that if $n = 9$, we can always find such 3 points.

  (c) Find a set of $n = 8$ points, where it is impossible to choose 3 such points.

(4)
  (a) Show the following identity by a combinatorial argument:

$$S(n+1, k+1) = \sum_i \binom{n}{i} S(i, k)$$

  (b) The Bell numbers are defined by $B(n) := \sum_k S(n, k)$. What do they count? Find a bijection which proves that the number of partitions of $[n]$, such that no two consecutive numbers appear in the same block, is $B(n-1)$.

(5) In the parliament of some country there are $2n + 1$ seats filled by 3 parties. How many possible distributions (i, j, k), (i.e. party 1 has i, party 2 has j, and party 3 has k seats) are there, such that no party has an absolute majority? Give a nice combinatorial proof. (Hint: Look at small numbers and make a good guess.)

(6) $c(n, k)$ denotes the number of permutations of $[n]$ with k cycles. Show the following identity:

$$c(n+1, m+1) = \sum_{k=m}^{n} \binom{k}{m} c(n, k)$$