2. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 20. April 2021
Due dates: 27.-29. April
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI21.html`

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(1) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \quad \left[\text{Hint: } \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}\right]$$

(2) A permutation $\pi \in S_n$ is a transposition if exactly the position of two elements is switched, i.e. in cycle notation there are $n-2$ fixpoints and a cycle of length 2.

a) Show that each permutation $\pi \in S_n$ is a composition of transpositions. We denote the minimum number of transpositions needed to express π by $t(\pi)$.

b) As in the lecture, $c(\pi)$ denotes the number of cycles of π. Show that

$$t(\pi) + c(\pi) = n$$

(c) What is the expected number of left-to-right-maxima in a random permutation? What is the expectation of $t(\pi)$?

(3) Educate yourself about the Combinatorial Nullstellensatz. Find a different proof for a proposition in the lecture using the Combinatorial Nullstellensatz.

(4) There are nine caves in a forest, each being the home of one animal. Between any to caves there is a path, but they do not intersect. Since it is election time, the candidating animals travel through the forest to advertise themselves. Each candidate visits every cave, starting from their one cave and arriving there at the end of the tour again. Also each path is only used once in total, since nobody wants to look at the embarrassing posters (not even their own), put there on the first traverse. Under these assumptions, what is the maximum possible number of candidates?

(5) Mutually orthogonal latin squares (MOLS)

(a) Prove one part of the unproved theorem from the lecture: Construct $(n-1)$ MOLS of order n from a projective plane of order n.

(b) Let $\mathbb{F}$ be a field of n elements. For all $q \in \mathbb{F}\backslash\{0\}$, define $n \times n$ tables Q_q by $Q_q(x, y) = qx + y$. Show that the tables of this family are MOLS.

(6) A *projective plane* is a collection of points and lines, such that

- For any two points there is exactly 1 line containing both of them.
- Any two lines intersect in exactly 1 point.
- There are 4 points such that no line passes through more than 2 of them.

(a) Prove that if l is a line and p a point not on l, then the number of points on l is the same as the number of lines through p.

(b) Prove that any finite projective plane has the same number of points and lines.