Basic Counting

Rule of sum: A, B finite $A \cap B = \emptyset$

$\Rightarrow |A \cup B| = |A| + |B|$

Rule of Product: A, B finite

$\Rightarrow |A \times B| = |A| \cdot |B|$

Rule of Exponentiation: A, B finite

$\Rightarrow |A^B| = |A|^{|B|}$

$A^B = \{f \mid f: B \to A\}$

Rule of Bijection: $\exists$ bijection $A \to B$

$\Rightarrow |A| = |B|$

Proving identities combinatorially

Find two ways of counting the elements of a set $\mathcal{A}$

Examples:

1. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$\mathcal{A}$ = k-subsets of $[n]$ $\quad x \in [n] \; / \; \mathcal{A}_x + \mathcal{A}^x$

- $\binom{n}{k} \cdot k = n \binom{n-1}{k-1}$

$(A, a) \longleftrightarrow (a, A')$

$a \in A \quad \boxed{A = A' + a} \quad a \notin A'$

Binomial Coefficients

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{(n)_k}{k!}$$

$\binom{S}{k}$ = k-subsets of S

Lemma $\binom{n}{k} = \left|\binom{[n]}{k}\right|$

Proof

- k profile of a permutation

$(\pi_1 \dots \pi_n) \to (b_1 \dots b_n)$

$$b_i = \begin{cases} 1 & \pi_i \leq k \\ 0 & \pi_i > k \end{cases}$$

k profile is a 0,1-vector of length n with k entries 1

each such vector comes from $k!(n-k)!$ different permutations

0,1-vect of length n with k entries 1 $\overset{bij}{\longleftrightarrow}$ k subsets of $[n]$

$$\Rightarrow \quad n! = \left|\binom{[n]}{k}\right| \cdot k!(n-k)! \qquad \square$$

Yet another model for k subsets of $[n]$: path in grid from $[0,0] \to [n-k,k]$ with steps up-right.

Pascals Triangle

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \end{array}$$

- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$

$$\begin{array}{ccc} n & m & k \\ A & B & C \\ \multicolumn{3}{c}{A \supseteq B \supseteq C} \end{array} \Longleftrightarrow \begin{array}{ccc} n & k & X \\ A & C & \boxed{A \setminus C \quad B \setminus C} \\ \multicolumn{2}{c}{A \subseteq C} & X \subseteq A \setminus C \\ && |X| = m-k \end{array}$$

- $\sum_{l=0}^{k} \binom{n}{l}\binom{m}{k-l} = \binom{n+m}{k}$ Vandermonde
- $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$

$\boxed{\sum \binom{n+k}{n} = \binom{n+m+1}{n+1}}$

Extended binomial coefficients

$(r)_k = r(r-1)\dots(r-k+1)$
is a polynomial in r can be evaluated $\forall r \in \mathbb{C}$

$\Rightarrow \binom{r}{k}$ makes sense for $r \in \mathbb{C}$ $k \in \mathbb{N}$

Ex: $\binom{-r}{k} = \frac{-r(-r-1)(-r-2)\dots(-r-k+1)}{k!}$

$= (-1)^k \frac{(r+k-1)_k}{k!} = (-1)^k \binom{r+k-1}{k}$

in particular

$\binom{-1}{k} = (-1)^k \binom{1+k-1}{k} = (-1)^k$

What about extending identities?

- Symmetry $\binom{n}{k} = \binom{n}{n-k}$ only makes sense for $n-k \in \mathbb{N}$ $\Rightarrow n \in \mathbb{N}$

- $\binom{x}{m}\binom{m}{k} = \binom{x}{k}\binom{x-k}{m-k}$

  both sides are polynomials in x of degree m
  They agree in $> m$ evaluations $\Rightarrow$ polynomials agree.

- Vandermonde

$$\sum_{\ell=0}^{k} \binom{x}{\ell}\binom{y}{k-\ell} = \binom{x+y}{k}$$

Thm: If $p, q \in K[x,y]$ $\deg p, \deg q \leq m$ agree on all $(s_i, t_j) \in S \times T$ with $(S, T \subseteq K)$ $|S| = |T| = m+1$ $\Rightarrow p = q$

Proof: $V = \langle x^k y^\ell : 0 \leq k, \ell \leq m \rangle$

$\dim V \leq (m+1)^2$

Let $p_{k\ell}(s_i, t_j) = \begin{cases} 1 & k=i, \ell=j \\ 0 & \text{else} \end{cases}$

$\Rightarrow \exists a_{k\ell}$ st. $\Rightarrow p_{k\ell}(x,y) = a_{k\ell} \prod_{k \neq i} (x - s_k) \prod_{\ell \neq j} (y - t_\ell)$

$\Rightarrow p_{k\ell} \in V$

$\{p_{k\ell}\}$ lin indep if $\sum c_{k\ell} p_{k\ell} = 0$

$\Rightarrow$ eval at $(s_k t_\ell)$ shows that $c_{k\ell} = 0$

$\Rightarrow \{p_{kl}\}$ is a basis of $V = K_{\leq m}[x,y]$

if p,q are as in the statement

$\Rightarrow p,q \in V$, $(p-q) = \sum \lambda_{kl} p_{kl}$

eval at (s_k, t_l) shows $\lambda_{kl} = 0$

$\Rightarrow p - q \equiv 0$ □

From Vandermonde we obtain (multiply by $k!$)

$$(x+y)_k = \sum_{l=0}^{k} \binom{k}{l} (x)_l (y)_{k-l}$$

Binomial Theorem

$$\boxed{(x+y)^k = \sum_{l=0}^{k} \binom{k}{l} x^l y^{k-l}}$$

Proof Idea I

Expand left side and compare coefficient of $x^k y^{k-l}$

(crucial: commutativity)

what if $A, B \in R$: $BA = qAB$?

Proof Idea II

$$(X \cup Y)^K = \sum_{L \subseteq K} X^L \times Y^{K \setminus L}$$

this yields $(a+b)^k = \sum \binom{k}{l} a^l b^{k-l}$

for infinitely many pairs (a,b)

$\Rightarrow$ the polynomials are the same □

Some counting related to permutations

We have seen representations of a permutation
- two line
- one line
- cycle repr.

Notation $\pi \in S_n$

$c_i = c_i(\pi) = \#$ cycles of length i in π

$\text{type}(\pi) = (c_1 \ldots c_n)$

$\#(\text{cycles of } \pi) = c(\pi) = c_1 + c_2 + \ldots + c_n$

Proposition For $c = (c_1 \ldots c_n)$ with $c_i \geq 0$, $\sum k c_k = n$

there are $\dfrac{n!}{1^{c_1} c_1! \, 2^{c_2} c_2! \cdots n^{c_n} c_n!}$ permutations of type c in S_n

Proof.

- A map f_c from permutations to permutations of type c
  add set of parantheses

Ex $c = (1, 2, 1, 0 0 0 0)$

$\pi = 3 7 1 5 8 4 6 2 \longrightarrow (3)(7 1)(5 8)(4 6 2)$

$\pi' = 3 8 5 7 1 2 4 6 \longrightarrow (3)(8 5)(7 1)(2 4 6)$

Not injective

Question: How many perm. have the same image?

- Can permute cycles of same length
- Can choose any element of a cycle as the cycle leader (first)

$\square$

Let $c(n,k)$ = # permutations in S_n with k cycles ($\sum c_i = k$)

Rem: $s(n,k) = (-1)^{n-k} c(n,k)$ are stirling numbers of first kind

Lemma (recursion)

$$c(n,k) = c(n-1,k-1) + (n-1)\,c(n-1,k)$$

and $c(n,k) = 0$ when $n \le 0$ or $k \le 0$

exception $c(0,0) = 1$

$n \backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	11	6	1	
	24	50	35	10	1

special cases

$c(n,1) = (n-1)!$

$c(n,n) = 1$

$c(n,n-1) = \binom{n}{2}$

Proof of Lemma: use cycle notation

- n a cycle on its own $\pi \to \pi - n$
- n has a predecessor
  $\pi \to (p, \pi - n)$

Thm: $\sum_{k=1}^{n} c(n,k)\, x^k = x^{\underline{n}}$ raising factorial

$x^{\underline{n}} = x(x+1)\dots(x+(n$

Proof I Use the recursion

Let $F_n(x) = \sum_{k=1}^{n} c(n,k)\, x^k$

$$F_n(x) = \sum_k c(n,k)\, x^k =$$
$$= \sum c(n-1,k-1)\, x^k + \sum (n-1)\, c(n-1,k)\, x^k$$
$$= x F_{n-1}(x) + (n-1) F_{n-1}(x)$$
$$= (x+(n-1)) F_{n-1}(x)$$

Induction

$F_1(x) = x \qquad F_2(x) = x + x^2 = x(x+1)$ $\square$

Proof II show equality for $x \in \mathbb{N}$

$\Rightarrow$ polynomial identity

LHS

$\sum c(n,k)\, x^k$ counting pairs (π, f) where $\pi \in S_n$ and f maps each cycle z of π to $1 \le f(z) \le x$ an integer

RHS

counting

$(b_1, \dots, b_n)$

with $1 \le b_i \le x+i-1$

A bijection $(b_1 \dots b_n) \longmapsto (\pi, f)$ given in canonical cycle notation • Leaders largest • increasing Leaders

• n starts a cycle z with $f(z) = b_1$

i elements

when $\boxed{n \;\ldots\; n-i+1}$ have been used
consider b_{i+1}

Case I $\quad 1 \le b_{i+1} \le x$

place $n-i$ at beginning of the partial permutation
$n-i$ will be Left-to-right min, cycle leader of γ
and define $f(\gamma) = b_{i+1}$

Case II $\quad b_{i+1} = x + k \qquad 1 \le k \le i$

insert $n-i$ behind k already existing elements in the word

Example $\quad x = 3 \quad n = 8$

	$3 \ge b_1$	$4 \ge b_2$	$5 \ge b_3$	$6 \ge b_4$	$7 \ge b_5$	$8 \ge b_6$	$9 \ge b_7$	$10 \ge b_8$
$\boxed{\text{Case I} \;\le 3}$	2	1	5	3	7	4	1	5
Case II k:			2		4	1		2

① ③ ① ②

(8)
(7 8)
(7)(8 6)
(5)(7)(8,6)
(5)(7)(8 6 4)
(5 3)(7)(8 6 4)
(2)(5 3)(7)(8 6 4)
$\underbrace{(2)}_{1}\underbrace{(5\,1\,3)}_{5}\underbrace{(7)}_{1}\underbrace{(8\,6\,4)}_{2}$

cycle leaders are the largest elements of their cycles

cycles are ordered with increasing cycle leaders

reverse: 3 7 1 5 8 4 6 2
② ① ③

$1 \mapsto 2+x$
$2 \mapsto 6+x$
$3 \mapsto$ ②
$4 \mapsto 3+x$

$5 \mapsto 1+x$
$6 \mapsto 2+x \qquad x=3$
$7 \mapsto$ ①
$8 \mapsto$ ③

(3 1 5 4 6 2 9 5)

5

Expected number of cycles

Let A be a k subset of $[n]$

Def $X_A(\pi) = \begin{cases} 1 & \pi \text{ has a } k\text{-cycle whose elements are the } k \text{ elements of } A \\ 0 & \text{otherwise} \end{cases}$

$$Pr(X_A = 1) = \frac{1}{n!} \left(\# \pi \text{ having a } k \text{ cycle whose elem are } A \right)$$

$$= \frac{(k-1)!\,(n-k)!}{n!} = \frac{1}{k} \binom{n}{k}^{-1}$$

Def $Z_k(\pi) = C_k(\pi) = \# \, k$ cycles

Note: $Z_k(\pi) = \sum_{A \in \binom{[n]}{k}} X_A(\pi)$

$$E(Z_k) = E(\sum X_A) = \sum E(X_A)$$

$$= \sum Pr(X_A = 1) = \sum_{A \in \binom{[n]}{k}} \frac{1}{k} \binom{n}{k}^{-1}$$

$$= \binom{n}{k} \frac{1}{k} \binom{n}{k}^{-1} = \frac{1}{k}$$

Def $Z(\pi) = \sum Z_k(\pi)$

$$E(Z) = E(\sum Z_k) = \sum_{k=1}^{n} \frac{1}{k} = H_n \sim \ln(n)$$

harmonic number

The Twelvefold Way

We count functions

$f: N \longrightarrow M$ $\quad |N| = n$

Balls $\quad$ Boxes (Kistchen) $\quad |M| = m$

we may restrict f to be

- injective $(m \geq n)$
- surjective $(n \geq m)$
- arbitrary (no restriction)

In many cases it is interesting to count equivalence classes

eg: $\binom{m}{n}$ $\boxed{\begin{array}{l} n \text{ indistinguishable balls} \\ m \text{ distinguishable boxes} \\ \text{injective} \end{array}}$

Def: Indistinguishable boxes

$f \sim g \iff \exists \pi \in S_M$ such that

$\pi \circ f = g$

Indistinguishable balls

$f \sim g \iff \exists \sigma \in S_N$ such that

$f = g \circ \sigma$

the table

balls	boxes	arbitr	inj $(m \geq n)$	surj $(n \geq m)$
D	D	1) m^n	2) $(m)_n$	3) $m!\,S(n,m)$
I	D	4) $\binom{n+m-1}{m-1}$	5) $\binom{m}{n}$	6) $\binom{n-1}{m-1}$
D	I	7 $\sum_{t \leq m} S(n,t)$	8 1	9 $S(n,m)$
I	I	10 $\sum_{t \leq m} P_t(n)$	11 1	12 $P_m(n)$

$S(n,m)$ Stirling number of 2nd kind.

$P_m(n)$ Partition of n into m parts

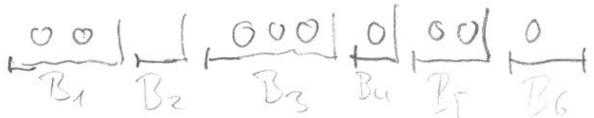
Comments 8,11 siehe Skizze in Tabelle

n balls
m boxes

1, 2 ✓

456 Ball i Boxes / borders

4)



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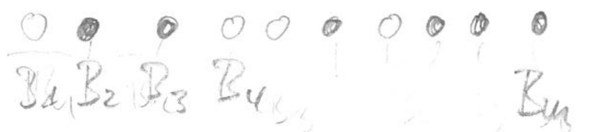

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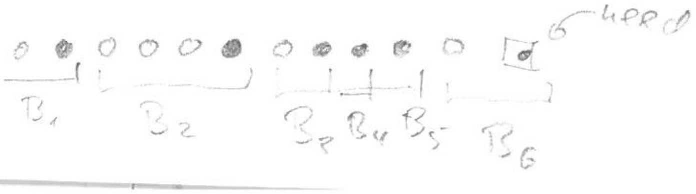

③ Partitions of a set N

$$P = \{S_1, S_2, \dots, S_m\}$$

$S_i \neq \emptyset$; $\bigcup S_i = N$; $S_i \cap S_j = \emptyset$

$S(n,m)$ = # Partitions of a n set into m blocks

$S(0,0) := 1$ $\quad S(n,n) = 1$ $\quad S(n,n-1) = \binom{n}{2}$

$S(n,0) = 0 \quad n > 0$

$S(n,1) = 1$

$S(n,2) = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$

$$S(n,m) = m\,S(n-1,m) + S(n-1,m-1)$$

Proof consider element n

either singleton block

$\curvearrowright$ $P - n \in S(n-1,m-1)$

or not

$\curvearrowright$ $P - n \in S(n-1,m)$

can put n back in m different ways

	m=0	m=1	m=2	m=3	m=4	
n=0	1					
n=1	0	1				
n=2	0	1	1			
n=3	0	1	3	1		
	0	1	7	6	1	
	0	1	15	25	10	1

Obs: Entries (3, 7, 8) in the table follow easily

Prop: $x^n = \sum_{k=0}^{n} \binom{x}{k} k!\, S(n,k) = \sum S(n,k)(x)_k$

Vorsicht nicht unterstr.

❗ mit $t \in \mathbb{N}$ könnten wir statt $\sum^n$ auch $\sum^t$ schreiben, das macht bei auswertungen mit $x \in \mathbb{C}$ aber wenig Sinn, deshalb $\sum^n$ oder $\sum_{k \geq 0}$

Proof $\# f : [n] \to [t] = t^n$

choose $\mathrm{Im}(f) = Y \subseteq [t]$ if $|Y| = k$

$\exists\, k!\, S(n,k)$ surjective fkt

(⇒ equal as polynomial) ⇒

Stirling inversion

We have: $\boxed{x^n = \sum_{k=0}^{n} S(n,k)(x)_k}$ (1)

The stirling numbers of the first kind are $c(n,k) = \#\ \pi \in S_n$: π decomposes into k cycles

$s(n,k) = (-1)^{n-k} c(n,k)$

$n! = \sum c(n,k)$

Hatten wir 2017 schon

more general

$x^{\underline{n}} = \sum c(n,k) x^k$

See exercises.
We will see a nice proof in the context of Polya theory

From this with

$\hat{S}(n,k) = (-1)^{n-k} c(n,k)$ we get

Rüdi

$\boxed{(x)_n = \sum_{k=0}^{n} \hat{S}(n,k) x^k}$ Dual to (1)
(2)

From (1) and (2) we see that Matrices

$\left[S(n,k)\right]_{n,k \le N}$ and $\left[\hat{S}(n,k)\right]_{n,k \le N}$

Are transformation matrices for the bases $\{x^k\}_{k \le N}$ and $\{(x)_k\}_{k \le N}$ of $\mathbb{R}_{\le N}[x]$

This implies

- $\sum_{i \ge 0} S(m,i)\hat{S}(i,k) = \delta_{[m=k]}$ Inverses
- For sequences (a_k) and (b_k)

$$b_n = \sum_{k=0}^{n} S(n,k) a_k \iff a_n = \sum_{k=0}^{n} \hat{S}(n,k) b_k$$