13. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 3. July 2019
Due dates: 9./11. July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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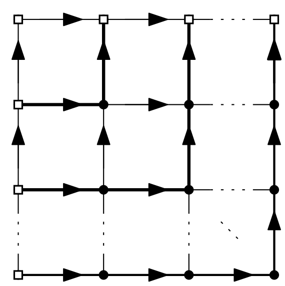
(1) Let $D = (\{v_1, \ldots, v_n, s, t\}, \vec{E})$ be a directed graph. We are interested in the number of *Hamilton paths* from s to t, that is, the number of paths from s to t that contain all the $v_1, ..., v_n$ as inner vertices exactly once. Develop a formula which can be evaluated in $o(n!)$ time. [Hint: You can use Möbius inversion.]

(2) Count the number of triples (p_0, p_1, p_2), where p_i are vertex-disjoint lattice paths of length 6 from $A_i = (i, 2-i)$ to $B_i = (3+i, 5-i)$.

(3) Apply the Lemma of Lindström-Gessel-Viennot in order to re-prove the formula for the Vandermonde determinant.
[Hint: Find appropriate weights for the edges of the 'lattice'-graph indicated below with its unique vertex disjoint paths system.]



(4) In the lecture we saw a lower bound on the number of perfect matchings of 3-regular bipartite multi-graphs. Show that this lower bound is essentially best-possible:

(a) Let $\mathcal{G}$ be the set of all 3-regular bipartite multi-graphs with $2n$ vertices, where the 2 partition classes have a linear order of their vertices and for every vertex, there is a linear order of the incident edges. How many such graphs are there?

(b) Let M be one of the $n!$ perfect matchings of the complete bipartite graph $K_{n,n}$. Count the number of graphs $G \in \mathcal{G}$ which contain M as a perfect matching.

(c) Conclude that no lower bound of c^n for $c > \frac{4}{3}$ can be shown.

(5) Let F_n denote the nth Fibonacci number. Use Kasteleyn signatures to find matrices $M(n)$ with

$$\det(M(n)) = F_n.$$

[Hint: Recall the model of 1-2-sums for the Fibonacci numbers.]

(6) A magician has cards numbered from 1 to 100, distributed in 3 boxes of different colors so that no box is empty. He/she lets one person of the crowd choose two cards from different boxes without watching. Then the person tells the magician the sum of the numbers on the two cards and he/she announces from which box no card was taken. In how many ways can the magician distribute the cards so that his trick always works?