11. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 18. June 2019
Due dates: 25./27. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1) (This exercise gives 2 points.) For which of the parameter sets does a design exist? Either show that there is none or present one. (if λ is omitted, it is 1.)

(a) $S(2,3,127)$ (d) $S(2,7,36)$

(b) $S_2(4,7,13)$ (e) $S_{10}(2,5,9)$

(c) $S_3(2,10,25)$ (f) $S_3(3,5,21)$

[Hint to (f): Consider the edge set of K_7.]

(2) Let $(\mathcal{P},\mathcal{B})$ be a $S_\lambda(t,k,v)$ design.

(a) Let $p \in \mathcal{P}$ and $\mathcal{B}^p := \{B : p \notin B \in \mathcal{B}\}$ be the set of blocks, which do not contain p. Show that $(\mathcal{P}\setminus\{p\},\mathcal{B}^p)$ is a design. What are its parameters?

(b) Consider the *complement* of a $S_\lambda(t,k,v)$, i.e., replace each block by its complement. Prove that the complement of a $S_\lambda(t,k,v)$ is a design for the same parameter t. Determine its other parameters.

(3) A design $(\mathcal{P},\mathcal{B})$ is *resolvable*, if there is a partition of $\mathcal{B}$, such that each part is a partition of $\mathcal{P}$.
Let $(\mathcal{P},\mathcal{B}) = S(2,n+1,n^2+n+1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that

$$\Big(\mathcal{P}\setminus B, \big\{C\setminus B \mid C \in \mathcal{B}\setminus\{B\}\big\}\Big)$$

is a resolvable $S(2,n,n^2)$ design.

(4) Let $(V,\mathcal{B})$ be a design, $I,J \subseteq V$ with $I\cap J=\emptyset$ and $|I|=i, |J|=j$ such that $i+j\le t$. Let $\lambda_{I,J} = \#\{B\in\mathcal{B} \mid I\subseteq B \text{ and } J\cap B=\emptyset\}$.

(a) Show that $\lambda_{I,J}$ does only depend on i and j and not on I and J, i.e. $\lambda_{i,j} := \lambda_{I,J}$ is well defined.

(b) Prove $\lambda_{i,j} = \lambda_{i+1,j} + \lambda_{i,j+1}$ for $i+j<t$.

(c) Prove $\lambda_{i,j} = \sum_{r=0}^{j}(-1)^r\binom{j}{r}\lambda_{i+r,0}$.

(5) Let q be a prime power. For every $k,n\in\mathbb{N}, k\le n$, construct the following design:

$$S_\lambda\left(2, \begin{bmatrix} k \\ 1 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q\right) \text{ with } \lambda = \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$