10. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 11. June 2019
Due dates: 18./20. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1) Consider necklaces with 12 beads of at most three different colors
   (a) How many different necklaces exist up to equivalence by the dihedral group?
   (b) How many different necklaces with 3 red, 4 green, and 5 blue beads exist?

(2) Consider red-blue-face-colorings of the platonic solids with rotational symmetries.
   (a) Determine the number of differently colored dodecahedra.
   (b) Determine the number of differently colored icosahedra.
   (c) Determine the number of differently colored soccer balls. Note that a soccer ball is a truncated icosahedron.

(3) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a relabeling of the vertices of G yields the graph H, i.e., if there exists a bijection $\Phi : V \to V'$ such that the mapping $\Psi(\{v, w\}) := \{\Phi(v), \Phi(w)\}$ is a bijection from E to E'.
Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges. Let S_n be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i, j\}) := \{\pi(i), \pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_{S_n}\Big(1 + x, 1 + x^2, \dots 1 + x^{\binom{n}{2}}\Big).$$

(4) Let there be 27 cubes of volume 1. Count the number of ways to paint the faces of the cubes with 3 colors, such that for any color c, you can stick the cubes together to form a cube of volume 27 whose faces are of color c, up to rotational symmetries of the 27 cubes it consists of.

(5) Popular matchings
Let G be a bipartite graph and each vertex v has a strict preference list L_v on its neighbors with additional minimal element $\emptyset$. Let M and M' be two matchings of G. For a vertex v, let $M(v)$ be its matched edge, or $\emptyset$ if it is not matched and

$$vote_v(M, M') = \begin{cases} -1, & v \text{ prefers } M'(v) \text{ over } M(v) \\ 0, & v \text{ has no prefered matching, e.g.,} M(v) = M'(v) \\ 1, & v \text{ prefers } M(v) \text{ over } M'(v) \end{cases}$$

We say M is *more popular* than M' if $\sum_v vote_v(M, M') \geq 0$ and write $M' \preceq M$. We say M is popular, if $M' \preceq M$ for all matchings M' of G.
   (a) Show that the relation 'more popular' is not acyclic.
   (b) Show that every stable matching is popular.
   (c) Show that popular matchings may be strictly larger than stable matchings.
   (*) Finding max-size popular matchings can be done by a modified Gale-Shapely. Inform yourself about the algorithm. (A rough idea of the proof is fine.)