9. Practice sheet for the lecture: Combinatorics (DS I)

Felsner/ Schröder

4. June 2019

Due dates: 11./13. June

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

---

(1) Theorem of Hall

(a) Find an infinite counterexample to the Theorem of Hall, i.e., find a bipartite graph $G = (X \cup Y; E)$ with the property that $|N(S)| \geq |S|$ for all $S \subset X$ and there is no matching containing all vertices of X.

(b) A bipartite graph is *biregular* if the vertices in X, Y have degree d_x and d_y, respectively. Show that biregular bipartite graphs $(X \cup Y, E), |E| \neq 0$ always allow for matchings of size $\min\{|X|, |Y|\}$.

(c) The analogue of the Hall condition for general graphs is the *Tutte condition.* Inform yourself about this condition and show at least one implication.

(2) Let $P = (X, \leq)$ be a poset. We call a chain decomposition $\{C_i\}_i$ of P *greedy chain decomposition (GCD)* if it has the following property: C_1 is a maximum chain in P, and for $i > 1$, C_i is a maximum chain in P_i where P_i is the subposet of P induced by $X - \bigcup_{j<i} C_j$. Prove or disprove: $\exists c \in \mathbb{R}$ such that any GCD has size at most $c \cdot w$, where w is the size of a minimum chain decomposition.

(3) Consider two magicians M_1, M_2 in well separated rooms. A volunteer picks five cards from a standard deck (52 cards) and hands them to M_1. M_1 keeps one of the five cards and puts the other four (in specific order) in an envelope. The envelope is brought to M_2 who opens it, has a look at the cards and announces the fifth card.

(a) Explain the existence of a strategy for this trick with the aid of Hall's Theorem.

(*) Find a playable strategy (which you can demonstrate with a colleague).

(4) Consider the poset P_n on the set $\left\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil}, b_1, \ldots, b_{\lfloor \frac{n}{2} \rfloor}\right\}$ with the cover relations $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $b_i > a_{i-1}$ as well as $a_i > b_{i-2}$ for all i. Count the linear extensions of P_n. [Hint: Use the same recursion as in the next exercise]

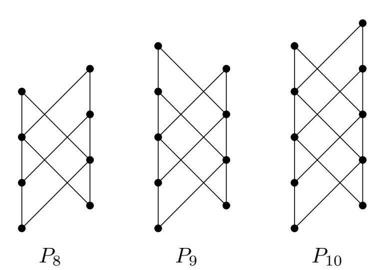

Figure 1: Hasse diagrams of P_8, P_9 and P_{10}

(5) Let $(P, \leq)$ be a *tree–shaped poset,* i.e., a poset such that for each $x \in P \setminus \min(P)$ there is a parent $y \in P$ with $z \leq y < x$ for all $z \in P$ with $z < x$. Consider a recursive procedure to count the linear extensions of $(P, \leq)$. Use this procedure to derive an explicit formula involving the hook $h(x) := |\{z \in P : x \leq z\}|$ of the elements of P.