8. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 28. Mai 2019
Due dates: 04./06. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1) Symmetric chain decompositions of $\mathcal{B}_n$

(a) Given a symmetric chain C in $\mathcal{B}_n$, is there always a symmetric chain decomposition containing C?

(b) Show that the number of chains of length $n+1-2k$ in a symmetric chain decomposition of $\mathcal{B}_n$ is $\binom{n}{k} - \binom{n}{k-1}$.

(c) Use (1b) to derive the known explicit formula for the Catalan numbers, i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$.
(Hint: Valid bracket expressions are counted by Catalan numbers.)

(2) Let $\mathcal{B}_n^{\vee}$ be the truncation of the Boolean lattice where the maximal and minimal element is deleted. Let $\mathcal{C}$ be a symmetric chain decomposition which is canonical (originating from the bracketing process). Let $\overline{\mathcal{C}}$ be its complement, i.e. for a chain $C \in \mathcal{C}$ the set $\overline{C}$ of complements of sets in C is a set in $\overline{\mathcal{C}}$.

(a) Show that $\overline{\mathcal{C}}$ is a symmetric chain decomposition.

(b) Show that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are *orthogonal*, i.e. $|C \cap D| \leq 1$ for all $C \in \mathcal{C}$ and $D \in \overline{\mathcal{C}}$.

(3) Let $\mathcal{A}$ be a family of k-subsets of $[n]$ and $\mathcal{B}$ be a family of l-subsets of $[n]$ such that $l+k \leq n$ and $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) Show that $|\mathcal{A}| < \binom{n-1}{k-1}$ or $|\mathcal{B}| \leq \binom{n-1}{l-1}$.
Hint: Use shadows as in the second proof of the Erdős-Ko-Rado theorem.

(b) Deduce the Erdős-Ko-Rado theorem from (a).

(4) A permutation $\pi \in S_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \dots$ holds. Let $\mathrm{Alt}_n \subseteq S_n$ be the set of alternating permutations. A permutation σ is *reverse alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \dots$ holds. Let $\mathrm{RAlt}_n \subseteq S_n$ be the set of reverse alternating permutations.

(a) Prove $|\mathrm{Alt}_n| = |\mathrm{RAlt}_n|$.

(b) Let $E_n := |\mathrm{Alt}_n|$ and prove $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for all $n \geq 1$.

(c) Let $E_n(q) := \sum_{\pi \in \mathrm{RAlt}_n} q^{\mathrm{inv}(\pi)}$ and $E_n^{\star}(q) := \sum_{\pi \in \mathrm{Alt}_n} q^{\mathrm{inv}(\pi)}$. Prove

$$E_n^{\star}(q) = q^{\binom{n}{2}} E_n\left(\frac{1}{q}\right).$$

(5) Counting chains and antichains in $\mathcal{B}_n$

(a) What is the number $C_n^{(k)}$ of chains of size k in the boolean lattice?

(b) Prove by bijection that the number of antichains of size 2 is $\frac{1}{2}C_n^{(3)}$.

(c) What is the number of antichains of size 3 in the boolean lattice?