7. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 21. May 2019
Due dates: 28.-30. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1) Let $P := (M, \leq)$ be a finite poset with n elements, i.e. $|M| = n$. For a given ordering of the elements $(m_1, \ldots, m_n)$, we can associate with P a relation matrix $(a_{ij}) = A \in \{0,1\}^{n\times n}$, such that $a_{ij} = 1 \Leftrightarrow m_i \leq m_j$.

(a) Show that it is possible to find an ordering on $(m_1, \ldots, m_n)$ such that A is an upper triangle matrix.

(b) Show that $\#\{(i,j)|a_{ij} = 1\} = \binom{n+1}{2} \Leftrightarrow P$ is a total order.

(c) Find matrix properties for a $\{0,1\}$–matrix B, which imply that B represents a partial order relation?

(2) A poset $P = (X, \leq)$ is *ranked* if there exist a function $r : X \to \mathbb{N}$ such that for all $x \in X$, the rank $r(x)$ is the length of *all* maximal chains ending in x.
For $n \in \mathbb{N}$, the *divisor–poset* P_n is the set of all divisors of n ordered by divisibility:

$$P_n := \big\{\{x \in \mathbb{N} : x \mid n\}, \{(x,y) \in \mathbb{N}^2 : x \mid y \text{ and } y \mid n\}\big\}.$$

Prove that P_n is ranked. Sketch the Hasse–Diagramm of P_{60} and compute its ranks. Is P_n a lattice?

(3) q-analogue

(a) How many different maximal chains of nested subspaces of $V_n(q)$ exist?

(b) Define the q-analogue of the boolean lattice. Consider an antichain $\mathcal{A}$ and show that the q–analogue of the LYM inequality holds:

$$\sum_{k=0}^{n} \frac{p_k(\mathcal{A})}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1$$

(4) Let $k \in \mathbb{N}$ be fix. Prove that for each $n \in \mathbb{N}$ there are unique $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ with $a_i \in \mathbb{N}$, such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$

(5)

(a) Show that every finite poset is isomorphic to the containment order on some family of sets.

(b) Show that every finite poset can be obtained by taking a subset of the integers $I \subseteq \mathbb{N}$ together with the divisor relation $|$.