6. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 15. May 2019
Due dates: 21.-23. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

---

(1) Prove the following identity:

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$$

(2) The q–binomials fulfill the equation

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

for all $n \geq k \geq 0$. Prove this via the lattice path model for q–binomials.

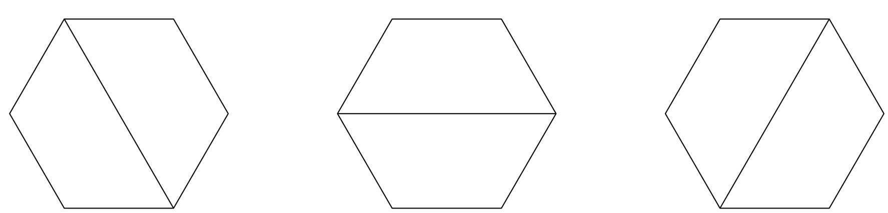
(3) Search in the literature for at least three different combinatorial structures that are also counted by Catalan numbers, one of which (at least) is not from Wikipedia, and prove that they are. (You are allowed to present proofs from the literature you read.)

(4) Let $\mathcal{B}$ be a set with a certain weight function $|\cdot| : \mathcal{B} \to \mathbb{N}$ and let $\mathcal{A}$ be the set of all finite subsets of $\mathcal{B}$. Show that the generating function A of $\mathcal{A}$ can be expressed as

$$A(z) = \prod_{n\geq 1} (1+z^n)^{b_n}$$

Recall that $A(z) = \sum_{a\in\mathcal{A}} x^{|a|} = \sum_{n\geq 0} a_n x^n$ and, likewise, $B(z) = \sum_{b\in\mathcal{B}} x^{|b|} = \sum_{n\geq 0} b_n x^n$. Think about the right extension of the weight for $a \in \mathcal{A}$.

(5) A set of chords of a convex $2n$-gon is a *quadrangulation* if no two chords intersect and all faces are quadrangles. Let a_n denote the number of quadrangulations of a convex $2n$-gon. Use the symbolic method to find the generating function $A(x) = \sum_{n\geq 0} a_n x^n$. The figure shows the 3 quadrangulations of a 6-gon:



[Hint: Find a connection to ternary trees.]