5. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Schröder**

7. May 2019

Due dates: 14.-16. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1) Let $S(n,k)$ denote the stirling numbers of the second kind. The Bell numbers are defined by

$$B(n) := \sum_{k=0}^{n} S(n,k)$$

(a) What does $B(n)$ count? Show the identity by a combinatorial argument:

$$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$$

(b) Let $F(z) := \sum \frac{B(n)}{n!} z^n$ be the exponential generating function of the Bell numbers $B(n)$. Show that there exists a function $f(z)$ such that $F'(z) = f(z)F(z)$.

(c) Solve the differential equation and deduce a closed formula for $F(z)$.

(2) On a table there are 100 coins. A and B are going to remove coins from the table by turns. In each turn they can remove 2, 5 or 6 coins. The first one that cannot make a move loses. Determine who has a winning strategy if A plays first.

(3) Let a_k denote the number of words of length k over the alphabet $\{u, l, r\}$ with no l and r consecutive, i.e. lr and rl do not appear. These words can be interpreted as grid paths of length k which go up, left and right, and do not intersect themselves.

(a) Find a linear recursion for a_k.

(b) Express the generating function as a rational function.

(c) Find a closed form for a_k.

(d) Give a combinatorial proof for the equation:

$$A(z) = \left(1 + 2\sum_{k=1}^{\infty} z^k\right) \cdot (z \cdot A(z) + 1)$$

(4) Prove the following facts about the Fibonacci numbers F_n.

(a) Every $n \in \mathbb{N}$ can be partitioned uniquely into different Fibonacci numbers $F_k, k > 1$, such that no two consecutive Fibonacci numbers are used.

(b) $F_{2n} = \sum_{k=1}^{n} \binom{n}{k} F_k$ [Hint: There is a nice bijection using monominoes.]

(5) You have three types of stamps, two different types with a value of 2 cent and one type with a value of 3 cent. Now you have to put stamps with a total value of k cent on an envelope. Let h_k be the number of feasible sequences of stamps. Find a closed form for h_k.