4. Practice sheet for the lecture: **Felsner/ Schröder**

Combinatorics (DS I) 30. April 2019

Due dates: 7.-9. May

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1) Show that the number of partitions of n where no part is divisible by $d \in \mathbb{N}\setminus\{0,1\}$ equals the number of partitions of n where no d parts have the same size.

(2) Consider walks in the plane starting in $(0,0)$ where each step is $R : (x,y) \to (x+1,y)$ or $U_a : (x,y) \to (x, y+a)$ with a a positive integer. There are 5 walks that contain a point on the line $x+y=2$, namely: $RR, RU_1, U_1R, U_1U_1, U_2$. Let a_n denote the number of walks that contain a point on the line $x+y=n$. Express a_n in terms of Fibonacci numbers. [Hint: Look at small numbers and make a good guess.]

(3) For fixed $s \in \mathbb{N}$, find a recursion for the sequence $(a_n(s))_{n\geq 0}$ where

$$a_n(s) = (1+\sqrt{s})^n + (1-\sqrt{s})^n.$$

(4) In how many ways can you pay n Dollar with 1$, 5$ and 10$ notes? Find a generating function and compute the number of ways to pay 50 Dollar.

(5)

(a) Let $g(n)$ be the number of subsets of $[n]$ that contain no two consecutive elements, for integer n. Find a recurrence that is satisfied by these numbers.

(b) Prove the following identity for the Fibonacci numbers F_n by bijection:

$$F_{n+2} + \sum_{k=2}^{n} 2^{n-k} F_{k-1} = 2^n$$

(6) Consider a tower of size $2 \times 2 \times n$ and bricks of size $2 \times 1 \times 1$. How many different tilings of the tower with bricks (and rotated copies) exist?
[Hint: Consider both, the number of tilings a_k of a tower of height k and the number of unfinished tilings b_k, where in the top level only 2 cubes (of the 4) are covered by standing bricks.]