3. Practice sheet for the lecture: **Felsner/ Schröder**
Combinatorics (DS I) 23. April 2017
Due dates: 30. April/2. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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(1)
  (a) How many subsets of the set $[n]$ contain at least one odd integer?
  (b) How many multi-subsets of the set $[n]$ have size k? (Multi(-sub-)sets may contain the same element more than once.)
  (c) Let $\subseteq$ denote the (non-strict) subset relation. For a given $k \in [n]$, how many sequences $(T_1, T_2, \ldots, T_k)$ are there with

$$\emptyset \subseteq T_1 \subseteq T_2 \subseteq \ldots \subseteq T_k \subseteq [n] \;?$$

(2) Let there be n points in the plane with integer coordinates. Among them, we want to find three points p_1, p_2, p_3, such that their barycenter $b = \frac{1}{3}(p_1 + p_2 + p_3)$ has integer coordinates.
  (a) Prove that if $n \geq 13$, we can always find such 3 points.
  (b) Prove that if $n = 9$, we can always find such 3 points.
  (c) Find a set of $n = 8$ points, where it is impossible to choose 3 such points.

(3) Let H_n denote the n-th harmonic number. Show the following identities:
  (a)

$$\sum_{1 \leq k < n} H_k = nH_n - n$$

  (b)

$$\sum_{1 \leq k < n} kH_k = \binom{n}{2} H_n - \frac{\binom{n}{2}}{2}$$

(4) (Balkan Mathematical Olympiad 2002) Suppose that, in a group of people, everyone knows at least three of the others. Show that you can choose an even number of them and seat them at a round table, such that everyone knows his two neighbours.
Remark: If you choose only two of them, nobody has the chance to know two of his neighbours, as they only have one. So you will have to choose more than two of them.

(5) $c(n, k)$ denotes the number of permutations of $[n]$ with k cycles. Show the following identity:

$$c(n+1, m+1) = \sum_{k=m}^{n} \binom{k}{m} c(n, k)$$