**2. Practice sheet for the lecture:** **Felsner/ Schröder**
**Combinatorics (DS I)** 16. April 2019
Due dates: 23.-25. April
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI19.html`

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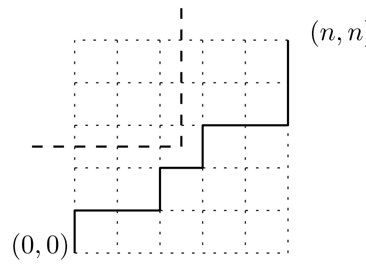
(1) Let $n \in \mathbb{N}$ be odd. Count the number of grid paths from $(0,0)$ to $(n,n)$ with 'up' and 'right' steps, which avoid all grid points $(i,j)$, such that $i > \frac{n}{2} > j$. The figure shows a path for $n = 5$.




(2) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \qquad \left[\text{Hint: } \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}\right]$$

(3) A permutation $\pi \in S_n$ is a transposition if exactly the position of two elements is switched, i.e. in cycle notation there are $n-2$ fixpoints and a cycle of length 2.

a) Show that each permutation $\pi \in S_n$ is a composition of transpositions. We denote the minimum number of transpositions needed to express $\pi$ by $t(\pi)$.

b) As in the lecture, $c(\pi)$ denotes the number of cycles of $\pi$. Show that

$$t(\pi) + c(\pi) = n$$

(4) Mutually orthogonal latin squares (MOLS)

(a) Invert the proof of the lecture: Construct $(n-1)$ MOLS from a projective plane of order $n$.

(b) Let $\mathbb{F}$ be a field of $n$ elements. For all $q \in \mathbb{F}\backslash\{0\}$, define $n \times n$ tables $Q_q$ by $Q_q(x,y) = qx + y$. Show that the tables of this family are MOLS.

(5) (Colombia 2011) Ivan and Alexander write lists of integers. Ivan writes all the lists of length $n$ with elements $a_1, a_2, \ldots, a_n$ such that $|a_1| + |a_2| + \cdots + |a_n| \leq k$. Alexander writes all the lists with length $k$ with elements $b_1, b_2, \ldots, b_k$ such that $|b_1| + |b_2| + \cdots + |b_k| \leq n$. Prove that Alexander and Ivan wrote the same number of lists.

(6) Prove the proposition from the lecture: For every projective plane there is $n \in \mathbb{N}$ (called the order of the plane), such that

- Every point lies in exactly $n+1$ lines.
- Every line contains exactly $n+1$ points.
- There are exactly $n^2 + n + 1$ points and $n^2 + n + 1$ lines.