11. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 30. June 2015
Due dates: 07.-09. July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) Let $A_1, \ldots, A_r \subseteq [n]$. Prove the following *inclusion-exclusion formula*:

$$\left|\bigcup_{i=1}^{r} A_i\right| = \sum_{I\subseteq[r], I\neq\emptyset} (-1)^{|I|-1}\left|\bigcap_{i\in I} A_i\right|$$

(a) by Möbius inversion.

(b) directly.

(2) How many positive integers less than $n \in \mathbb{N}$ have no factor between 2 and 10? How many are these if $n = 1000$?

(3) For a prime power q, consider the poset P of all subspaces of the n-dimensional vector space $V_n(q)$ over $\mathbb{F}_q$ with the subspace relation.

(a) Compute the Möbius function of P.

(b) Count the number of linear functions from $\mathbb{F}_q^n$ onto $\mathbb{F}_q^k$.

[Observe that $\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} (-1)^i q^{\binom{k}{2}} = 0$ follows from 'our' q-binomial theorems.]

(4) Use Möbius inversion to show that for every positive integer n, it holds

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$$

[Hint: Recall that $\sum_{d|n} \phi(d) = n$.]

(5) Count the number of triples (p_0, p_1, p_2), where p_i are vertex-disjoint lattice paths of length 6 from $A_i = (i, 2-i)$ to $B_i = (3+i, 5-i)$.

(6) Apply the Lemma of Lindström-Gessel-Viennot in order to

(a) show that for two matrices $A, B \in \mathbb{K}^{n\times n}$: $\det(A\cdot B) = \det(A)\cdot\det(B)$

(b) re-prove the formula for the Vandermonde determinant.
[Hint: Find appropriate weights for the edges of the 'lattice'-graph indicated below with its unique vertex disjoint paths system.]