10. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 23. June 2015
Due dates: 30. June-2. July
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) For which of the parameter sets does a design exist? Either show that there is none or present one. (This exercise gives 2 points.)

(a) $S(2,5,125)$ (d) $S(2,7,36)$

(b) $S_2(4,7,13)$ (e) $S_3(2,4,20)$

(c) $S(2,6,16)$ (f) $S_3(3,5,21)$

[Hint to (f): Consider K_7 and its following subgraphs with 5 edges: a star, a cycle, a triangle and two disjoint edges.]

(2) Let $(\mathcal{P},\mathcal{B})$ be a $S_\lambda(t,k,v)$ design.

(a) Let $p \in \mathcal{P}$ and $\mathcal{B}^p := \{B : p \notin B \in \mathcal{B}\}$ be the set of blocks, which do not contain p. Show that $(\mathcal{P} \setminus \{p\}, \mathcal{B}^p)$ is a design. What are its parameters?

(b) Let $J \subset \mathcal{P}$ with $|J| = j$. Show that the number of blocks of the $S_\lambda(t,k,v)$ design that are disjoint with J is

$$b^j = \lambda \frac{\binom{v-j}{k}}{\binom{v-t}{k-t}}$$

(c) Consider the *complement* of $S_\lambda(t,k,v)$, i.e., replace each block by its complement. Prove that the complement of $S_\lambda(t,k,v)$ is a t-design. Determine its parameters.

(3) Let $(\mathcal{P},\mathcal{B}) = S(2,n+1,n^2+n+1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that $(\mathcal{P} \setminus B, \{C \setminus B \mid C \in (\mathcal{B} \setminus \{B\})\})$ is a resolvable $S(2,n,n^2)$ design.

(4) Let $(V,\mathcal{B})$ be a design, $I,J \subseteq V$ with $I \cap J = \emptyset$ and $|I| = i, |J| = j$ such that $i+j \leq t$. Let $\lambda_{I,J} = \#\{B \in \mathcal{B} \mid I \subseteq B \text{ and } J \cap B = \emptyset\}$.

(a) Show that $\lambda_{I,J}$ does only depend on i and j and not on I and J, i.e. $\lambda_{i,j} := \lambda_{I,J}$ is well defined.

(b) Prove $\lambda_{i,j} = \lambda_{i+1,j} + \lambda_{i,j+1}$ for $i+j < t$.

(c) Prove $\lambda_{i,j} = \sum_{r=0}^{j} (-1)^r \binom{j}{r} \lambda_{i+r,0}$.

(5) Let q be a prime power. For every $k,n \in \mathbb{N}, k \leq n$, construct the following design:

$$S_\lambda\left(2, \frac{q^k-1}{q-1}, \frac{q^n-1}{q-1}\right) \text{ with } \lambda = \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

(*) In the lecture we saw two (isomophic) STS(15).

(a) Give another construction of an STS(15) by considering the edge set of K_6, together with edge sets forming triangles and perfect matchings.

(b) Show that it is isomorphic to the one from the lecture.