9. Practice sheet for the lecture: Combinatorics (DS I)

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) Euler Phi-function

(a) Prove that for m, n with $\gcd(m, n) = 1$ it holds that $\phi(m \cdot n) = \phi(m) \cdot \phi(n)$.

(b) For a prime p and an integer k, determine $\phi(p^k)$.

(2) How many necklaces of length 12 with beads of at most three different colors exist? Necklace equivalence is given by the dihedral group.

(3) Consider the platonic solids with all their rotational symmetries. We want to count different colorings of the faces with red or blue.

(a) Determine the number of differently colored dodecahedra.

(b) Determine the number of differently colored icosahedra.

(c) Determine the number of differently colored soccer balls. Note that a soccer ball is a truncated icosahedron.

(4) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a re-labeling of the vertices of G equals H, i.e. if there is a bijection $\Phi : V \to V'$ such that the mapping $\Psi((v, w)) := (\Phi(v), \Phi(w))$ is a bijection from E to E'.

(a) Count non–isomorphic graphs on four vertices applying a theorem of Polya.

(b) Count non–isomorphic graphs on four vertices when loops are allowed. *Loops* are edges starting and ending at the same vertex.

(c) Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges (no loops). Let G be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i, j\}) := \{\pi(i), \pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_G\Big(1 + x, 1 + x^2, \dots 1 + x^{\binom{n}{2}}\Big).$$

(5) A *rooted ternary tree* is a rooted tree where every vertex has at most 3 children. We do not distinguish between two rooted ternary trees if one can be obtained from the other by permuting the subtrees of its vertices.
Let $\mathcal{T}$ be the family of rooted ternary trees and G the symmetric group S_3 with its standard action on three elements. Let $\mathcal{F} = \mathcal{T}^{[3]}$ and consider the action of G on $\mathcal{F}$. For $f \in \mathcal{F}$, we define $w(f) = w(T_a) \cdot w(T_b) \cdot w(T_c)$, where $w(T) = x^{\#\text{ vertices of T}}$. For an orbit F of $\mathcal{F}$, we define $w(F) = w(f)$ for some $f \in F$.

(a) Determine $\sum_{F \text{ orbit}} w(F)$.

(b) Let t_n denote the number of rooted ternary trees on n vertices and denote its generating function by $T(x) := \sum t_n x^n = 1+x+x^2+2x^3+4x^4+8x^5+17x^6+\dots$. Conclude that

$$T(x) = 1 + x \cdot P_G\Big(T(x), T(x^2), T(x^3)\Big).$$