8. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 09. June 2015
Due dates: 16.-18. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1)
(a) Prove that a graph G is bipartite if and only if there is no odd cycle in G. An odd cycle is a sequence of vertices and edges $v_1 e_1 v_2 e_2 \dots e_{2k} v_n e_{2k+1}$, such that $e_i = (v_{i-1}, v_i)$ and $e_{2k+1} = (v_{2k+1}, v_1)$ with $e_i \in E$ and $v_i \in V$.

(b) Let G be a graph and M be a matching in G. Color all edges e of G blue if $e \in M$ and red otherwise. The vertex v is *exposed* if all adjacent edges are red, i.e., do not belong to the matching. A path between two vertices is *alternating* if the colors of its edges are alternating in red and blue.
Show that a matching is maximum if and only if for all pairs of exposed vertices v, w there is no alternating path.

(2)
(a) Show that biregular bipartite graphs $(X \cup Y, E)$ always allow for matchings of size $\min\{|X|, |Y|\}$. A *biregular graph* is a graph where the nodes in X, Y have degree d_x and d_y, respectively.

(b) Show that a regular (i.e. all vertices have the same degree) bipartite graph can be covered with perfect matchings, i.e. that the set of edges can be partitioned into perfect matchings. Give a lower bound for the number of covers with perfect matchings.

(3) Consider two magicians M_1, M_2 in well separated rooms. A volunteer picks five cards from a standard deck (52 cards) and hands them to M_1. M_1 keeps one of the five cards and puts the other four (in specific order) in an envelope. The envelope is brought to M_2 who opens it, has a look at the cards, mumbles, and announces the fifth card.

(a) Explain the existence of a strategy for this trick with the aid of Hall's Theorem.

(*) Find a playable strategy (which you can demonstrate with a colleague).

(4) Find an infinite counterexample to the Theorem of Hall, i.e. find a bipartite graph $G = (X, Y; E)$ with the property that $|N(S)| \geq |S|$ for all $S \subset X$ such that there is no matching of size $|X|$.

(5)
(a) Let $(P, \leq)$ be a poset, consisting of n disjoint chains of length $a_1, a_2, \dots, a_n$. How many linear extensions does P have?

(b) Let $(P, \leq)$ be a poset and $\max((P, \leq)) := \{x \in P \mid x \leq y \Rightarrow y = x\}$ be the set of its maxima. Let $e((P, \leq))$ be the number of linear extensions of $(P, \leq)$.Prove

$$e((P, \leq)) = \sum_{x \in \max(P)} e((P \setminus \{x\}, \leq')),$$

where $\leq'$ is the restriction of $\leq$ to $P \setminus \{x\}$, i.e. $\leq' := \leq \cap (P \setminus \{x\}) \times (P \setminus \{x\}) \subseteq P \times P$.

(6) Consider the poset P_n on the set $\left\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil}, b_1, \ldots, b_{\lfloor \frac{n}{2} \rfloor}\right\}$ with the cover relations $a_i < a_{i+1}$, $b_i < b_{i+1}$, and $b_i > a_{i-1}$ as well as $a_i > b_{i-2}$ for all i. Count the linear extensions of P_n.

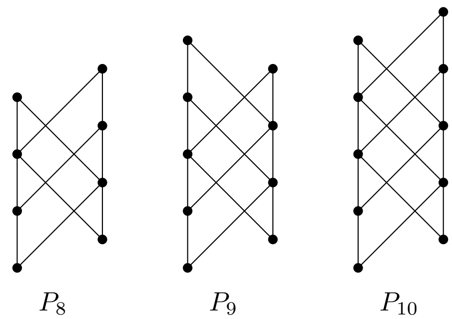

Figure 1: Hasse diagrams of P_8, P_9 and P_{10}