7. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 01. June 2015
Due dates: 09.-11. June
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) *Please hand in your solution of this exercise in a written form.* Let $k \in \mathbb{N}$ be fix. Prove by the colex-ordering of the k subsets, that for each $n \in \mathbb{N}$ there are unique $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ with $a_i \in \mathbb{N}$, such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$

(2) Symmetric chain decompositions of $\mathcal{B}_n$

(a) Given a symmetric chain C in $\mathcal{B}_n$, is there a symmetric chain decomposition containing C?

(b) Show that the number of chains of length $n - 2k$ in a symmetric chain decomposition of $\mathcal{B}_n$ is $\binom{n}{k} - \binom{n}{k-1}$.

(3) Symmetric chain decompositions of $\mathcal{B}_n$ and Catalan numbers

(a) Give a bijection between pairings of brackets of length $2n$ and binary trees with n nodes.

(b) Use (2b) and (3a) to derive the known explicit formula for the Catalan numbers, i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(4) Let $\mathcal{B}_n^{\vee}$ be the truncation of the Boolean lattice where the maximal and minimal element is deleted. Let $\mathcal{C}$ be a symmetric chain decomposition which is canonical (originating from the bracketing process). Let $\overline{\mathcal{C}}$ be its complement, i.e. for a chain $C \in \mathcal{C}$ the set $\overline{C}$ of complements of sets in C is a set in $\overline{\mathcal{C}}$.

(a) Show that $\overline{\mathcal{C}}$ is a symmetric chain decomposition.

(b) Show that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are *orthogonal*, i.e. $|C \cap D| \leq 1$ for all $C \in \mathcal{C}$ and $D \in \overline{\mathcal{C}}$.

(5) The following is an incorrect proof of Dilworth's Theorem. Find the mistake: Induction on $n := |P|$; $n = 1$ is obvious. For the induction step $n \to (n+1)$ let us assume the theorem holds for posets of n elements. By the *width* of P, we mean the size of a maximum antichain. Let $m \in P$ be a maximum of P. Apply the hypothesis to $P \setminus \{m\}$ and gain a decomposition into chains $C_1, \ldots, C_w$ of $P \setminus \{m\}$, with $w = \text{width}(P \setminus \{m\})$. If $w < \text{width}(P)$, then by adding $C_{w+1} = \{m\}$ as an additional chain, we gain a chain decomposition of P. If $w = \text{width}(P)$, the set $\{\max(C_i) \mid i = 1, \ldots, w\} \cup \{m\}$ cannot be an antichain. Therefore, $m \geq \max(C_i)$ for some $i = 1, \ldots, w$. Now $C_i \cup \{m\}$ is a chain, so $C_1, \ldots C_{i-1}, C_i \cup \{m\}, C_{i+1}, \ldots C_w$ is a chain decomposition of P. Thus, in any case, we have a chain decomposition of P with width(P) chains.

(6) Let $P = (X, \leq)$ be a poset. We call a chain decomposition $\{C_i\}_i$ of P *greedy chain decomposition (GCD)* if it has the following property: C_1 is a maximum chain in P, and for $i > 1$, C_i is a maximum chain in P_i where P_i is the subposet of P induced by $X - \bigcup_{j<i} C_j$. Quantify the (maximum) size of a greedy chain decomposition with respect to the size of a minimum chain partition.