6. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 19. May 2015
Due dates: 26.-28. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) Let $P := (M, \leq)$ be a finite poset with n elements, i.e. $|M| = n$. For a given ordering of the elements $(m_1, \ldots, m_n)$, we can associate with P a relation matrix $(a_{ij}) = A \in \{0,1\}^{n \times n}$, such that $a_{ij} = 1 \Leftrightarrow m_i \leq m_j$.

(a) Which conditions on $(m_1, \ldots, m_n)$ ensure, that A is an upper triangle matrix?

(b) Show that $\#\{(i,j)|a_{ij} = 1\} = \binom{n+1}{2} \Leftrightarrow P$ is a total order.

(c) Let k be the size of P's biggest chain. Show that A has the minimal polynomial $\mu_A(x) = (x-1)^k$.

(d) Find matrix properties for a $\{0,1\}$–matrix B, which imply that B represents a partial order relation?

(2) A poset $P = (X, \leq)$ is *ranked* if there exist a function $r : X \to \mathbb{N}$ such that for all $x \in X$, the rank $r(x)$ is the length of all maximal chains ending in x.
For $n \in \mathbb{N}$, the *divisor–poset* P_n is the set of all divisors of n ordered by divisibility:

$$P_n := \big\{\{x \in \mathbb{N} : x \mid n\}, \{(x,y) \in \mathbb{N}^2 : x \mid y \text{ and } y \mid n\}\big\}.$$

Prove that P_n is ranked. Sketch the Hasse–Diagramm of P_{60} and compute its ranks. Is P_n a lattice?

(3) Let $\mathcal{F}$ be an up-set of $\mathcal{B}_n$. Show, that the average size of the sets in $\mathcal{F}$ is at least $\frac{n}{2}$.

(4) q-analogues

(a) How many different maximal chains of nested subspaces of $V_n(q)$ exist?

(b) Show the q–analogue of the LYM inequality:

$$\sum_{k=0}^{n} \frac{p_k(\mathcal{A})}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1$$

(5) How many pairs (X, Y) of distinct subset of $[n]$ with $X \subset Y \subseteq [n]$ exist?

(a) Count the pairs.

(b) Give a generating function of the form $\frac{Q(z)}{P(z)}$ with polynomials P and Q.

(c) Prove the following equation bijectively:

$$\sum_{k=0}^{n} \binom{n}{k} (2^k - 1) = 3^n - 2^n$$

(6) Show that every finite poset is isomorphic to the containment order on some family of sets.

(7) Let A, B be up-sets and C, D down-sets of the boolean lattice $\mathcal{B}_n$.

(a) Prove $|A| \cdot |B| \leq 2^n \cdot |A \cap B|$ (Hint: Induction on n).

(b) Prove $|C| \cdot |D| \leq 2^n \cdot |C \cap D|$ (Hint: Apply (a)).

(c) Prove $|A| \cdot |C| \geq 2^n \cdot |A \cap C|$ (Hint: Apply (a)).

(This is known as Kleitman's Lemma)