5. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 11. May 2015
Due dates: 19.-21. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

(*) *We will start with excercises (6)-(8) of sheet 4. They count for this sheet.*

(1) The q–binomials fulfill the equation

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

for all $n \geq k \geq 0$. Prove this via the lattice path model for q–binomials. Can you give other proofs?

(2) Show that the q–binomials fulfill the equation

$$\sum_{k \geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} z^k = \prod_{i=0}^{n} \frac{1}{1-q^i z}$$

(3) Find and prove a q-analgue of the Vandermonde identity.

(4) The q–binomials fulfill the equation

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the model on $\mathbb{F}_q$ subspaces for q–binomials. Can you give other proofs?

(5) A permutation $\pi \in S_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \dots$ holds. Let $\mathrm{Alt}_n \subseteq S_n$ be the set of alternating permutations. A permutation σ is *reverse alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \dots$ holds. Let $\mathrm{RAlt}_n \subseteq S_n$ be the set of reverse alternating permutations.

- (a) Prove $|\mathrm{Alt}_n| = |\mathrm{RAlt}_n|$.
- (b) Let $E_n := |\mathrm{Alt}_n|$ and prove $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for all $n \geq 1$. [Hint: Apply (a)]
- (c) Let $E_n(q) := \sum_{\pi \in \mathrm{RAlt}_n} q^{\mathrm{inv}(\pi)}$ and $E_n^\star(q) := \sum_{\pi \in \mathrm{Alt}_n} q^{\mathrm{inv}(\pi)}$. Prove

$$E_n^\star(q) = q^{\binom{n}{2}} E_n\left(\frac{1}{q}\right).$$

(6) Let $F(n)$ be the number of walks in $\mathbb{Z}^2$, moving either one unit up, down, left or right, which start in $(0,0)$ and return to $(0,0)$ aftern n steps. Give a closed form for $F(n)$. There are many correct solutions; give a nice one!

(7) Suppose n persons are queued randomly such that each person is facing left or right. Every hour, all the persons who look into another persons face turn around by 180°. Is there a time when the process stops, i.e., there is no future movement? If so, how many hours does it take at most until the process stops?