**4. Practice sheet for the lecture:** **Felsner/ Kleist**
**Combinatorics (DS I)** 05. May 2015
Due dates: 12.-14. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) What can be said about the number of *horizontally convex polynominoes* of size $n$? Consider Example 14.5 of Lint & Wilson: A Course in Combinatorics, 2nd Edition, pp. 132. Prepare a 5-minute presentation.

(2) Inform yourself about the so called *Lagrange Inversion* in the context of generating functions. Prepare a 5-minute talk about your findings which also answers the following three questions:

(a) What is the Lagrange Inversion?

(b) State the idea of the proof.

(c) Why is this relevant? (Give examples.)

(3) Let $a_k$ be the number of words of length $k$ over the alphabet $\{u, l, r\}$ with no substring of the form $lr$ or $rl$. These words can be interpreted as grid paths of length $k$ which go up, left and right, and do not intersect themselves.

(a) Find a linear recurrence for $a_k$.

(b) Give a generating function for $a_k$, computed from the linear recursion.

(c) Find a closed form for $a_k$.

(d) Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ be the generating function for the words. Give a combinatorial proof for the equation

$$A(z) = \left(1 + 2\sum_{k=1}^{\infty} z^k\right) \cdot (z \cdot A(z) + 1)$$

and directly deduce a closed form for $A(z)$ without using (a).

(4) For fixed $s \in \mathbb{N}$, find a recursion for the sequence $(a_n(s))_{n \geq 0}$ where

$$a_n(s) = (1 + \sqrt{s})^n + (1 - \sqrt{s})^n.$$

(5) Consider a tower of size $2 \times 2 \times n$ and bricks of size $2 \times 1 \times 1$. How many different tilings of the tower with bricks (and rotated copies) exist?
[Hint: Consider both, the number of tilings $a_k$ of a tower of height $k$ and the number of unfinished tilings $b_k$, where in the top level only 2 cubes (of the 4) are covered by standing bricks.]

(6) In how many ways can you pay $n$ Dollar with 1\$, 5\$ and 10\$ notes? Find a generating function and compute the number of ways to pay 50 Dollar.

(7) Catalan numbers

(a) Complete the proof of the lecture for the Catalan numbers, i.e. simplify

$$c_n = \frac{1}{2}(-1)\binom{\frac{1}{2}}{n+1}(-4)^{n+1}.$$

(b) Use Lagrange Inversion in order to derive the Catalan numbers.

(8) Let $F(z) := \sum \frac{B(n)}{n!} z^n$ be the exponential generating function of the Bell numbers $B(n)$. Show that there exist a function $f(x)$ such that

$$F'(x) = f(x)F(x).$$

Solve the differential equation and deduce a closed formula for $F(x)$.