3. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 27. April 2015
Due dates: 05.-07. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) The Stirling numbers of first kind $s(n,k)$ count the number of permutations in S_n consisting of k disjoint cycles.

(a) Compute all values of $s(n,k)$ for $n \leq 4$ and give a (bijective) proof of the recurrence
$$s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1).$$

(b) Let $x^{\underline{n}}$ and $x^{\overline{n}}$ denote the falling and raising factorials. Prove the identities:
$$x^{\overline{n}} = \sum_{k=0}^{n} s(n,k)x^k$$
$$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)x^k$$
(Hint: Prove the first one inductively and deduce the second from the first).

(2)

(a) Show the following identity by a combinatorial argument:
$$S(n+1,k+1) = \sum_i \binom{n}{i} S(i,k)$$

(b) The Bell numbers are defined by $B(n) := \sum_k S(n,k)$. What do they count? Derive a recurrence for them from the identity given in (a) and interpret it combinatorially.

(c) Give a combinatorical argument, which proves that the number of partitions of $[n]$, such that no two consecutive numbers appear in the same block, is $B(n-1)$.

(3)

(a) How many subsets of the set $[n]$ contain at least one odd integer?

(b) For a given $k \in [n]$, how many sequences $(T_1, T_2, \ldots, T_k)$ are there with
$$\emptyset \subset T_1 \subset T_2 \subset \ldots \subset T_k \subseteq [n] \ ?$$

(4) *Please hand in your solution of this excercise in a written form:*
In the parliament of some country there are $2n+1$ seats filled by 3 parties. How many possible distributions (i,j,k), (i.e. party 1 has i, party 2 has j, and party 3 has k seats) are there, such that no party has an absolute majority? Give a nice combinatorial proof.
(Hint: Look at small numbers and make a good guess.)

(5) Number Partitions

(a) Prove that the number of partitions of n into at most k parts equals the number of partitions of $n+k$ into exactly k parts.

(b) The *Durfee square* of a partition P is the largest square fitting in the bottom left corner of P's Ferrers shape. Show that the number of partitions of n into exactly k different odd parts equals the number of self-conjugate partitions of n where the side length of the Durfee square is k.

(6) A composition of n is an ordered set of numbers $(a_1, \dots, a_k)$ with $a_i \in \mathbb{N}$ and $0 < a_i \leq n$ such that $a_1 + \cdots + a_k = n$. For example, $n = 4$ has the compositions: $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$, $3+1$, $1+3$, 4.

(a) Prove that n has $\binom{n-1}{k-1}$ compositions into exactly k parts/numbers.

(b) Let $c(n)$ be the number of compositions of n into an even number of even parts. For $n = 8$, these compositions are: $2+2+2+2$, $4+4$, $6+2$, $2+6$. Give a closed formula for $c(n)$.

(7) *Please hand in your solution of this excercise in a written form:*
In the lecture we saw a proof of Binet's formula based on generating functions. An alternative approach comes from linear algebra:

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \implies \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ where } A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Derive Binet's Formula by diagonalizing the matrix A.