2. Practice sheet for the lecture: Combinatorics (DS I)

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Due dates: 28-30. April
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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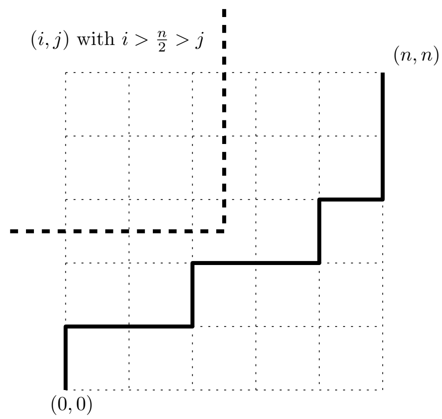
(1) Let $n \in \mathbb{N}$ be odd. Count the number of grid paths from $(0,0)$ to (n,n) with 'up' and 'right' steps, which avoid all grid points (i,j), such that $i > \frac{n}{2} > j$ (the figure below shows one of theses paths for $n = 5$).



(2) *Please hand in your solution of this exercise in a written form:*
Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials.* Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \quad \text{[Hint: } (-x)^{\overline{n}} = (-1)^n x^{\underline{n}}\text{]}$$

(3) Find a closed form expression for A_n, $n \in \mathbb{N}$.

$$A_n = \sum_{k=0}^{n} \binom{n-k}{k} (-1)^k.$$

(4) Let $\left(\!\binom{n}{k}\!\right)_g$ be the number of multisets with k elements (counted with multiplicity) of an n-element ground set, where each element occurs less than g times.
[Hint: Think about a characteristic vector model for multisets.]

(a) Show the Vandermonde identity:

$$\left(\!\binom{m+n}{k}\!\right)_g = \sum \left(\!\binom{m}{l}\!\right)_g \left(\!\binom{n}{k-l}\!\right)_g$$

(b) Show that

$$(1 + x + \cdots + x^{g-1})^n = \sum_{k=0}^{(g-1)n} \left(\!\binom{n}{k}\!\right)_g x^k.$$

(5) Prove the following identities (preferably by bijection!) for the Fibonacci numbers f_n.

(a) $$f_n + f_{n-1} + \sum_{k=0}^{n-2} 2^{n-k-2} f_k = 2^n$$

(b) $$f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1}$$

(6) Consider for $f : \mathbb{N} \to \mathbb{C}$ the operators

- $\Delta f(x) = f(x+1) - f(x)$ (Difference)
- $E^\alpha f(x) = f(x+\alpha)$ (Translation)
- $If(x) = f(x)$ (Identity)

The sequence f is an antiderivative of g if $\Delta f = g$. Show that:

(a) $\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E^k f(x)$. (Hint: $\Delta = E - I$)

(b) $\sum_{k=a}^{b} g(k) = f(b+1) - f(a)$ if f is the antiderivative of g.

(c) $\sum_{k=0}^{m-1} (k)_n = \frac{(m)_{n+1}}{n+1}$. (Hint: Use (b))