1. Practice sheet for the lecture: **Felsner/ Kleist**
Combinatorics (DS I) 13. April 15

Delivery date: 21. April

`http://www.math.tu-berlin.de/~felsner/Lehre/dsI15.html`

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(1) How many different words occur as permutation of the letters of ABRAKADABRA?

(2) The squares of a 4×7 chessboard are colored arbitrarily black and white (i.e. there are different $2^{4 \cdot 7}$ colorings). Show that there is an $i \times j$ rectangle with $i \geq 2$ and $j \geq 2$ such that all of its corners have the same color. Is the same true for a 4×6 chessboard?

(3) A sequence of numbers $a_1, a_2, \ldots, a_n$ is called *unimodal*, if there exists an $m \in [n]$, such that $a_i \leq a_{i+1}$ for all $i < m$ and $a_i \geq a_{i+1}$ for all $i \geq m$. Give three different proofs of the unimodality of the sequence $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ for all $n \in \mathbb{N}$, based on the three given hints:

(a) Use the definition

$$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!}.$$

(b) Consider the recursive definition

$$\binom{n}{k} := \binom{n-1}{k-1} + \binom{n-1}{k} \text{ and } \binom{n}{0} = \binom{n}{n} = 1,$$

based on Pascale's triangle.

(c) Use the bijection between $\binom{n}{k}$ and the number of subsets of $[n]$, having k elements.

(4) Permutations

(a) Give an alternative proof for the generating function of derangements (6)

$$D(z) := \sum \frac{d(n)}{n!} z^n = \frac{e^{-z}}{1-z}$$

by manipulating $D(z)e^z$ and using identity (2a) from the lecture.

(b) Prove of the following identity:

$$d(n) = n \cdot d(n-1) + (-1)^n$$

(c) How many permutations in S_n are derangements and involutions?
(Let X be a set. A function $f : X \to X$ is an *involution* if $f(f(x)) = x \; \forall x \in X$.)

(d) What is the expected number of fixed points of a uniform random permutation?

(5) Mutually orthogonal latin squares (MOLS)

(a) Complete the proof of the lecture: Construct $(n-1)$ MOLS from a projective plane of order n.

(b) Let $\mathbb{F}$ be a field of n elements. For all $q \in \mathbb{F} \backslash \{0\}$, define $n \times n$ tables Q_q by $Q_q(x, y) = qx + y$. Show that the tables of this family are MOLS.