(1)

(a) Given a symmetric chain C in $\mathcal{B}_n$, is there a symmetric chain decomposition containing C?

(b) Show that the number of chains of length $n-2k$ in a symmetric chain decomposition of $\mathcal{B}_n$ is $\binom{n}{k} - \binom{n}{k-1}$.

(2)

(a) Give a bijection between pairings of brackets of length $2n$ and binary trees with n nodes.

(b) Use (1b) and (2a) to derive the Catalan numbers.

(3) Consider the following two orders on the set of permuations S_n: Let $\pi_1, \pi_2 \in S_n$

- $\pi_1 \prec \pi_2$ if π_2 can be obtained by a transposition of two elements at adjacent positions $i, i+1$ with $\pi_1(i) < \pi_1(i+1)$. This is known as weak Bruhat order.

- $\pi_1 \leq \pi_2$ if π_2 can be obtained from π_1 by the transposition of two elements at positions i, j with $i < j$ and $\pi_1(i) < \pi_1(j)$. The transitive closure of these relations is known as the strong Bruhat order.

(a) Are they ranked?

(*) Do they have symmetric chain decompositions?

(4) Let $\mathcal{B}_n^\vee$ be the truncation of the Boolean lattice where the maximal and minimal element is deleted. Let $\mathcal{C}$ be a symmetric chain decomposition which is canonical (originating from the bracketing process). Let $\overline{\mathcal{C}}$ be its complement, i.e. with a chain $C \in \mathcal{C}$ the set $\overline{C}$ of complements of sets in C is a set in $\overline{\mathcal{C}}$. Show that $\overline{\mathcal{C}}$ is a symmetric chain decomposition. Show that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are *orthogonal*, i.e. $|C \cap D| \leq 1$ for all $C \in \mathcal{C}$ and $D \in \overline{\mathcal{C}}$.

(5) Let P be a finite poset with $\text{height}(P) = m$, where the *height* of a poset is the number of elements in a longest chain. Show that P can be partitioned into m antichains. (This is known as Anti-Dilworth theorem.)

(6) The following is an incorrect proof of Dilworth's Theorem. Find the mistake: Induction on $n := |P|$; $n = 1$ is obvious. For the induction step $n \to (n+1)$ let us assume the theorem holds for posets of n elements. Let $m \in P$ be a maxima of P. Apply the hypothesis to $P \setminus \{m\}$ and gain a decomposition into chains $C_1, \ldots, C_w$ of $P \setminus \{m\}$, with $w = \text{width}(P \setminus \{m\})$. If $w < \text{width}(P)$ then add $C_{w+1} = \{m\}$ as additional chain and we have a chain decomposition of P. If $w = \text{width}(P)$ the set $\{\max(C_i) \mid i = 1, \ldots, w\} \cup \{m\}$ can not be an antichain. Therefore $m \geq \max(C_i)$ for some $i = 1, \ldots, w$. Now $C_i \cup \{m\}$ is a chain, so $C_1, \ldots C_{i-1}, C_i \cup \{m\}, C_{i+1}, \ldots C_w$ is a chain decomposition of P. Thus in any case we have a chain decomposition of P with $\text{width}(P)$ chains.