7. Practice sheet for the lecture: **Felsner/ Kleist, Hoffmann**
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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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(1) *Please hand in your solution of this exercise in a written form.* Let $k \in \mathbb{N}$ be fixed. Prove, that for each $n \in \mathbb{N}$ there are unique $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ with $a_i \in \mathbb{N}$, such that

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}.$$

(2) In an office, at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's inbox. When there is time, the secretary takes the top letter of the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in order 1,2,3,4,5,6,7,8,9. While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such *after lunch typing orders* are possible? (That there are no letters left to be typed is one of the possibilities.) Generalize.

(3) A *linear extension* of a poset $(\{m_1, \ldots, m_k\}, \leq)$ is an extension of the partial order to a total order preserving the relations of the partial order, i.e. a total ordering $m_1 <' m_2 <' \cdots <' m_k$ such that there is no $i > j$ with $m_i \leq m_j$.
Consider the poset P_{i+j} on the set $\{a_1, \ldots, a_i, b_1, \ldots, b_j \mid j \in \{i-1, i\}\}$ with the relations $a_n < a_{n+1}$, $b_n < b_{n+1}$, and $b_n > a_{n-1}$ as well as $a_n > b_{n-2}$ for all n.
Count the number linear extensions of P_k.

(4) Let $P := (M, \leq)$ be a finite poset with n elements, i.e. $|M| = n$. A relation matrix $(a_{ij}) = A \in \{0,1\}^{n \times n}$ is matrix, such that $a_{ij} = 1 \Leftrightarrow m_i \leq m_j$ for some order $M = (m_1, m_2, \ldots, m_n)$.

(a) Which conditions on $(m_1, \ldots, m_n)$ suffice to ensure, that A is an upper triangle matrix?

(b) Show that

$$\#\{(i,j) | a_{ij} = 1\} = \binom{n+1}{2} \Leftrightarrow P \text{ is a total order.}$$

(c) Which conditions on a $\{0,1\}$–matrix B imply that B represents a partial order relation?

(d) Let k be the size of P's biggest chain. Show that A has the minimal polynomial $\mu_A(x) = (x-1)^k$.