6. Practice sheet for the lecture: Combinatorics (DS I) | **Felsner/ Kleist, Hoffmann** | 14. May '13

Delivery date: 21. – 22. May
http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html

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(1) The q–binomials fulfill the equation

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n-k \\ m-k \end{bmatrix}$$

for all $n \geq m \geq k \geq 0$. Prove this via the model on $\mathbb{F}_q$ subspaces for q–binomials. Can you give other proofs?

(2) How many different chains of length n of nested subspaces of $V_n(q)$ exist?

(3) Show the q–analogue of the LYM inequality:

$$\sum_{k=0}^{n} \frac{p_k(\mathcal{A})}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1$$

(4) How many pairs (X, Y) of distinct subset of $[n]$ with $X \subset Y \subseteq [n]$ exist?

(a) Count the pairs.

(b) Give a generating function of the form $\frac{Q(z)}{P(z)}$ with polynomials P and Q.

(c) Prove the following equation bijectively:

$$\sum_{k=0}^{n} \binom{n}{k} (2^k - 1) = 3^n - 2^n$$

(5) Show that every finite poset is isomorphic to the containment order on some family of sets.

(6) Let $\mathcal{B} \subseteq \binom{[n]}{k}$.

(a) Show the lemmata from the lecture:

$$|\triangledown \mathcal{B}| \geq \frac{n-k}{k+1} |\mathcal{B}| \qquad \text{for } k < n \text{ and}$$

$$|\vartriangle \mathcal{B}| \geq \frac{k}{n-k+1} |\mathcal{B}| \qquad \text{for } k > 0.$$

(b) Show Sperner's theorem using (a). Especially notice that $|\mathcal{B}| < |\triangledown \mathcal{B}|$ for $k < \frac{n-1}{2}$.