5. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Kleist, Hoffmann**

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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(1)

(a) Complete the lectures proof for the Catalan numbers, i.e. simplify

$$C_n = \frac{1}{2}(-1)\binom{\frac{1}{2}}{n+1}(-4)^{n+1}$$

(b) Use $a_n = \frac{A^{(n)}(0)}{n!}$ to obtain the Catalan numbers, where $A(x)$ is the generating function of (a_n).

(c) Use Lagrange Inversion in order to derive the Catalan numbers.

(2) Show that the q–binomials fulfill the equation

$$\sum_{k\geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix} z^k = \prod_{i=0}^{n} \frac{1}{1-q^i z}$$

(3) The q–binomials fulfill the equation

$$\sum_{i=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \cdot q^{(k+1)(n-i)} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

for all $n \geq k \geq 0$. Prove this via the lattice path model for q–binomials. Can you give other proofs?

(4) A permutation $\pi \in S_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \dots$ holds. Let $\mathrm{Alt}_n \subseteq S_n$ be the set of alternating permutations. A permutation σ is *reverse alternating* if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \dots$ holds. Let $\mathrm{RAlt}_n \subseteq S_n$ be the set of reverse alternating permutations.

(a) Prove $|\mathrm{Alt}_n| = |\mathrm{RAlt}_n|$.

(b) Let $E_n := |\mathrm{Alt}_n|$ and prove $2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$ for all $n \geq 1$ (Hint: Apply (a)).

(c) Let $E_n(q) := \sum_{\pi \in \mathrm{RAlt}_n} q^{inv(\pi)}$ and $E_n^\star(q) := \sum_{\pi \in \mathrm{Alt}_n} q^{inv(\pi)}$. Prove

$$E_n^\star(q) = q^{\binom{n}{2}} E_n\left(\frac{1}{q}\right).$$

(5) Ralf and Anna go to a dinner party with $n-1$ other couples. Each person shakes hands with everyone he or she doesn't know. Later, Ralf does a survey and discovers that everyone of the $2n-1$ other attendees shook hands with a different number of people. How many people did Anna shake hands with?