**3. Practice sheet for the lecture:** **Felsner/ Kleist, Hoffmann**
**Combinatorics (DS I)** 23. April '13
Delivery date: 29. April – 02. May
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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*Due to the 1st of May, there will be no tutorial session on tuesday, either. Please turn in your solutions to exercises (1) and (2) in a written form until thursday. Exercises (3), (4), (5) will be discussed the week after together with sheet 4.*

(1) Let $F(n)$ be the number of walks in $\mathbb{Z}^2$, moving either one unit up, down, left or right, which start in $(0,0)$ and return to $(0,0)$ aftern $n$ steps. Give a closed form for $F(n)$. There are many correct solutions; give a nice one!

(2)

(a) The *Durfee square* of a partition $P$ is the largest square fitting in the bottom left corner of $P$'s Ferrers shape. How can you determine the square's size directly from the partition without considering the Ferrers diagram?

(b) Give a proof, stating that the number of partitions of $n$ into at most $k$ parts is as big as the number of partitions of $n+k$ into exactly $k$ parts.

(c) Proof, that the number of partitions of $n$ into exactly $k$ different odd parts equals the number of self-conjugated partitions of $n$ where the side length of the Durfee square is $k$.

(d) Show, that each partition of $n$ has either at least $\sqrt{n}$ parts or the biggest part is $\geq \sqrt{n}$.

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(3)

(a) Show the following identity by a combinatorial argument:

$$S_{n+1,k+1} = \sum_i \binom{n}{i} S_{i,k}$$

(b) The Bell numbers are defined by $\text{Bell}(n) := \sum_k S_{n,k}$. What do they count? Derive a recurrence for them from the identity given in (a) and interpret it combinatorially.

(4) $n$ persons stand next to each other in a line. Now, they randomly turn either by 90° to the right or left side, such that there may exist pairs which stand face to face. All of these pairs turn by 180° at the same time. Is there a time when nobody turns anymore? If so, after how many time steps do they stop at most?

(5) In the parliament of some country there are $2n+1$ seats filled by 3 parties. How many possible distributions $(i,j,k)$, (i.e. party 1 has $i$, party 2 has $j$, and party 3 has $k$ seats) are there, such that no party has an absolute majority? Give a nice combinatorial proof.
(Hint: Look at small numbers and make a good guess.)