2. Practice sheet for the lecture: Combinatorics (DS I) **Felsner/ Hoffmann**
16. April

Delivery date: 24. April
`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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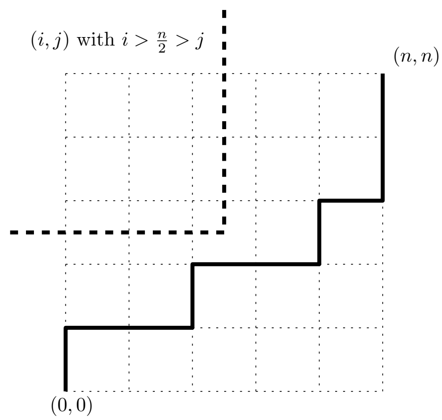
(1) Let $n \in \mathbb{N}$ be odd. Count the number of lattice paths from $(0,0)$ to (n,n), moving only up and right, which avoid all lattice points (i,j), such that $i > \frac{n}{2} > j$ (the figure below shows one of these paths for $n = 5$).



(2) Please hand your solution of this exercise in a written form in:

(a) Let $x^{\underline{n}} := (x)_n$ denote the falling factorials and $x^{\overline{n}} := x \cdot (x+1) \cdots (x+n-1)$ the *raising factorials*. Deduce the following equation from Vandermonde's identity:

$$(x+y)^{\overline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \quad \text{(Hint: } (-x)^{\overline{n}} = (-1)^n x^{\underline{n}}\text{)}$$

(b) The Stirling numbers of first kind $s(n,k)$ count the number of permutations in S_n consisting of k disjoint cycles. Compute all values of $s(n,k)$ for $n \le 4$ and give a (bijective) proof of the equation $s(n,k) = (n-1)s(n-1,k) + s(n-1,k-1)$.

(c) Prove the identities $x^{\overline{n}} = \sum_{k=0}^{n} s(n,k)x^k$ and $x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)x^k$ (Hint: prove the first one inductively and deduce the second from the first).

(3) Let $\left(\!\binom{n}{k}\!\right)_g$ be the number of multisets with k elements (counted with multiplicity) of an n-element ground set, where each element occurs less than g times. (Hint: Think about a characteristic vector model for multisets.)

(a) Show the Vandermonde identity:

$$\left(\!\!\binom{m+n}{k}\!\!\right)_g = \sum \left(\!\!\binom{m}{l}\!\!\right)_g \left(\!\!\binom{n}{k-l}\!\!\right)_g$$

(b) Show that

$$(1 + x + \cdots + x^{g-1})^n = \sum_{k=0}^{(g-1)n} \left(\!\binom{n}{k}\!\right)_g x^k.$$

(4) Consider for $f : \mathbb{N} \to \mathbb{C}$ the operators

- $\Delta f(x) = f(x+1) - f(x)$ (Difference)
- $E^\alpha f(x) = f(x+\alpha)$ (Translation)
- $If(x) = f(x)$ (Identity)

The sequence f is an antiderivative of g if $\Delta f = g$. Show that:

(a) $\Delta^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E^k f(x)$. (Hint: $\Delta = E - I$)

(b) $\sum_{k=a}^{b} g(k) = f(b+1) - f(a)$ if f is the antiderivative of g.

(c) $\sum_{k=0}^{m-1} (k)_n = \frac{(m)_{n+1}}{n+1}$. (Hint: Use (b))