11. Practice sheet for the lecture: Combinatorics (DS I) — **Felsner/ Kleist, Hoffmann**

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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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(1) For which of the parameter sets does a design exist? Either show that there is no design or present one. (This exercise gives 2 points.)

   (a) $S_2(4, 7, 13)$

   (b) $S(2, 7, 36)$

   (c) $S(2, 4, 13)$

   (d) $S(2, 6, 16)$

   (e) $S(2, 5, 125)$

   (f) $S(1, 4, 124)$

(2) Let $(V, \mathcal{B})$ be a $S_\lambda(t, k, v)$ design. Let $p \in V$ and $\mathcal{B}^p := \{B : p \notin B \in \mathcal{B}\}$ be the set of blocks, which do not contain p. Show that $(V \setminus \{p\}, \mathcal{B}^p)$ is a design. What are its parameters?

(3) Let $(V, \mathcal{B}) = S(2, n+1, n^2+n+1)$ be a projective plane and fix $B \in \mathcal{B}$. Show that $(V \setminus B, \{C \setminus B \mid C \in (\mathcal{B} \setminus \{B\})\})$ is a $S(2, n, n^2)$ design.

(4) Let $(V, \mathcal{B})$ be a design, $I, J \subseteq V$ with $I \cap J = \emptyset$ and $|I| = i, |J| = j$ such that $i + j \leq t$.. Let $\lambda_{I,J} = \#\{B \in \mathcal{B} \mid I \subseteq B \text{ and } J \cap B = \emptyset\}$.

   (a) Show that $\lambda_{I,J}$ does only depend on i and j and not on I and J, i.e. $\lambda_{i,j} := \lambda_{I,J}$ is well defined.

   (b) Compute all λ_{ij} for the $S_6(3, 5, 10)$ design from the lecture.

   (c) Prove $\lambda_{i,j} = \lambda_{i+1,j} + \lambda_{i,j+1}$ for $i + j < t$.

   (d) Prove $\lambda_{i,j} = \sum_{r=0}^{j} (-1)^r \binom{j}{r} \lambda_{i+r,0}$.

(5) Prove Fisher's inequality, which states that every $S_\lambda(t, k, v)$ design $(V, \mathcal{B})$ with $t \geq 2$ and $k < v$ fulfills $|V| \leq |\mathcal{B}|$ (Hint: Use the adjacency matrix $A \in \mathbb{R}^{|V| \times |B|}$ with $a_{v,B} = 1$ if $v \in B$ and $a_{v,B} = 0$ otherwise and consider the rank of $A \cdot A^T$).