10. Practice sheet for the lecture: **Felsner/ Kleist, Hoffmann**
Combinatorics (DS I) 11. June '13
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`http://www.math.tu-berlin.de/~felsner/Lehre/dsI13.html`

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(1) How many necklaces of length 12 with beads of at most three different colors exist? Count them modulo the dihedral symmetries.

(2) In this exercise we consider the plantonic solids with all their symmetries.

(a) Count the number of indistinguishable dodecahedra whose faces are colored with at most two colors.

(b) Count the number of indistinguishable icosahedra whose faces are colored with at most two colors.

(c) How many different soccer balls colored with at most two colors exist. Note that a soccer ball is a truncated icosahedron.

(3) A graph $G = (V, E)$ is *isomorphic* to a graph $H = (V', E')$, if a re-labeling of the vertices of G equals H, i.e. if there is a bijection $\Phi : V \to V'$ such that the mapping $\Psi((v, w)) := (\Phi(v), \Phi(w))$ is a bijection from E to E'.

(a) Count the number of non–isomorphic graphs with four vertices.

(b) Count the number of non–isomorphic graphs with four vertices and when loops are allowed. *Loops* are edges starting and ending in the same vertex.

(c) Let $g_{n,k}$ be the number of non-isomorphic graphs on n vertices with k edges. Let G be the symmetric group on the vertices, which acts on $\binom{[n]}{2}$ by $\pi(\{i, j\}) := \{\pi(i), \pi(j)\}$. Prove

$$\sum_{k=0}^{\binom{n}{2}} g_{n,k} x^k = P_G(1 + x, 1 + x^2, \dots 1 + x^{\binom{n}{2}}).$$

(4) (a) A spider has a sock and a shoe for each of its eight feet. In how many different ways can it put on its shoes and socks, assuming that on each foot it has to put on the sock first?

(b) Let $(P, \leq)$ be a poset, consisting of n disjoint chains of length $l_1, l_2, \dots, l_n$. How many linear extensions does P have?

(c) Let $(P, \leq)$ be a poset and $\max((P, \leq)) := \{x \in P \mid x \leq y \Rightarrow y = x\}$ be the set of its maxima. Let $e((P, \leq))$ be the number of linear extensions of $(P, \leq)$. Prove

$$e((P, \leq)) = \sum_{x \in \max(P)} e((P \setminus \{x\}, \leq')),$$

where $\leq'$ is the restriction of $\leq$ to $P \setminus \{x\}$, i.e. $\leq' := \leq \cap (P \setminus \{x\}) \times (P \setminus \{x\}) \subseteq P \times P$.

(5) Let P be a poset with a non-separating linear extension L. Show $\dim(P) \leq 2$.